A DIAGRAMMATIC METHOD FOR HANDLING INVARIANTS BUILT FROM 3-j SYMBOLS

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Diagrams corresponding to contracted products of 3-j symbols are introduced. There is exactly one invariant corresponding to each closed diagram. Identities for invariants follow from the rules for handling diagrams. Some illustrative examples are discussed.

1. Introduction

In the theory of atoms, nuclei, and elementary particles, one often meets invariants composed from 3-j symbols. For examples and references see *e.g.* Edmonds (1957). In this paper we describe a simple diagrammatic method of handling such invariants.

A 3-j symbol is denoted by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{1}$$

or sometimes (*cf.* *e.g.* Wigner 1959) by

$$(j_1 \cdot j_2 \cdot j_3 \cdot). \tag{1a}$$

It does not change if an even permutation of columns is performed *e.g.*

$$(j_1 \cdot j_2 \cdot j_3 \cdot) = (j_3 \cdot j_1 \cdot j_2 \cdot). \tag{2}$$

An odd permutation is equivalent to a multiplication by $(-1)^{j_1+j_2+j_3}$, *e.g.*

$$(j_1 \cdot j_2 \cdot j_3 \cdot) = (-1)^{j_1+j_2+j_3} (j_3 \cdot j_2 \cdot j_1 \cdot). \tag{3}$$

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The j's are interpreted as angular momenta, or spins, coupled by

$$\hat{j}_1+\hat{j}_2+\hat{j}_3 = 0, \tag{4}$$

and the m's as their projections on the z axis fulfilling consequently

$$m_1+m_2+m_3 = 0. \tag{4a}$$

Note that (4) implies that $(j_1+j_2+j_3)$ is an integer, and therefore that

$$(-1)^{2(j_1+j_2+j_3)} = +1, \tag{5}$$

as is necessary to make (3) consistent.

The transformation properties of 3-j symbols are well known (*e.g.* Wigner 1959). Symbol (1) is a tensor, covariant in each of the three indices m_1, m_2, m_3. In order to obtain the contravariant components one raises the indices using the formula

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} (-1)^{j-m} = \begin{pmatrix} j_1 & j_2 & m \\ m_1 & m_2 & j \end{pmatrix} = (j_1 \cdot j_2 \cdot j^{\cdot}). \tag{6}$$

We adopt the convention that if an expression contains both $j_i.$ and $j_i^{\cdot}$, the corresponding tensor indices are contracted. Thus expression (7) contains a six-fold sum. If both $j_i.$ and $j_i^{\cdot}$ are contained more than once, it must be stated which pairs should be coupled. As usually, contraction reduces the rank of the tensor. In particular we obtain an invariant if all the indices are contracted; in fact it follows from (6) that any invariant built from 3-j symbols can be written as such a fully contracted tensor, multiplied at most by a phase factor. For example

$$\begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{Bmatrix} = (j_1 \cdot j_2 \cdot j\,.)\ (j_1 \cdot j_4 \,.\, j'\,.)\ (j_3 \,.\, j_2 \,.\, j'\,\cdot)\ (j_3 \cdot j_4 \,.\, j\,.), \tag{7}$$

the 6-j symbol, is an invariant.

We shall also need the following relation between 3-j symbols

$$\sum_{j_3} (j_1 \,.\, j_2 \,.\, j_3\,.)\ (j_1' \cdot j_2' \cdot j_3 \cdot) \cdot (2j_3+1) = 1. \tag{8}$$

Here $j_i' = j_i$ and $m_i' = m_i$. The primes are added only to indicate that there is no contraction over m_1 and m_2.

2. *Diagrams*

To every invariant a diagram can be ascribed, according to the following prescription. For each 3-j symbol a vertex is drawn with three lines labelled with the j's coupled through this symbol. There should be one line for every pair of contracted j's. Thus each line joins two vertices. Such diagrams are described by Edmonds (1957).

We propose the following extension. Each line should be oriented from the vertex with $j^{\cdot}$ to the vertex with $j.$. Round each vertex the direction from the first to the second to the third j from the corresponding 3-j symbol should be marked. Because of the invariance with respect to cyclic permutations (2) only the direction $j_1 \to j_2 \to j_3 \to j_1$ is relevant. Which j is the first one is immaterial. It is easily seen that there is exactly one invariant corresponding to any given diagram. A diagram corresponding to invariant (7) is shown in Fig. 1.

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Let us note that according to (6) changing the direction of a line labelled j_i corresponds to the multiplication of the invariant by $(-1)^{2j_i}$. It is seen from formula (5), however, that changing the orientations of three lines coupled in one vertex leaves the invariant unchanged. Changing the direction of rotation around a vertex is equivalent, because of (3), to a multiplication by $(-1)^{j_1+j_2+j_3}$. If, however, there are two vertices coupling the same three momenta, reverting both orientations leaves the tensor unchanged. Thus in such cases only the relative orientation is important.

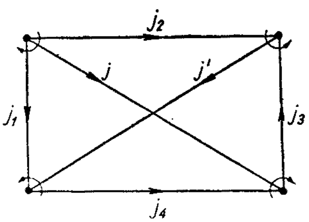
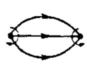
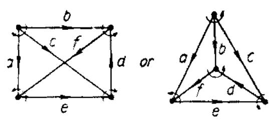
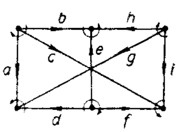
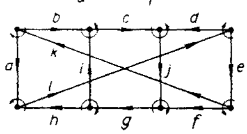
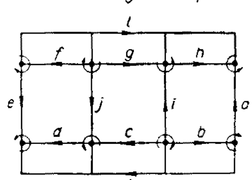


Fig. 1. Diagram corresponding to invariant (7)

TABLE I

Diagram	*Symbol*	*Name*
	1	1
	$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$	6-*j*[1]
	$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}$	9-*j*[1]
	$\begin{Bmatrix} a & b & k & f \\ h & i & g & d \\ l & c & e & j \end{Bmatrix}$	12-*j* (a)[2]
		12-*j* (b)[3]

In Table I the diagrams corresponding to a constant and to 6-*j*, 9-*j*, and 12-*j* symbols are assembled. Other diagrams containing no more than 12 lines can be reduced to these (*cf.* Sec. 4).

[1] Symbols discussed *e.g.* by Edmonds (1957).

[2] Symbol discussed by Jahn and Hope (1954) and by Ord-Smith (1954).

[3] Symbol discussed by Elliott and Flowers (1955).

3. *Open diagrams*

The diagrams described in the preceding section are closed; they have no external lines. Sometimes it may be convenient to split the invariant into a contracted product of two or more tensors. For instance evaluating invariant (7) one might work out

$$(j_1 \cdot j_2 \cdot j \cdot)\ (j_1 \cdot j_4 \cdot j \cdot')\ (j_3 \cdot j_2 \cdot j' \cdot), \tag{10}$$

first, and then contract the result with the remaining 3-j symbol. Using the prescription from the preceding section we can ascribe to tensor (10) a diagram which, however, is an open diagram with three external lines (Fig. 2).

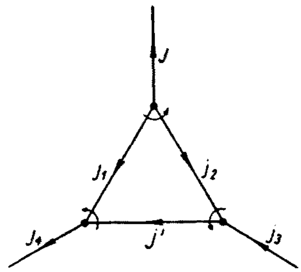


Fig. 2. Diagram corresponding to tensor (10)

We shall see that the contributions from open diagrams can often be easily evaluated. First, however, let us note the following

Lemma. For each open diagram

$$\sum m_e = 0, \tag{11}$$

where the summation is extended over the external lines, and all the m's are on one level, say covariant (lower) indices.

Indeed according to (3) the sum of m's for each vertex, and consequently also the sum of m's for all the vertices, is zero; but for each internal line the contributions from the two vertices cancel; therefore we are left with equality (11). Since (11) holds for an arbitrary orientation of the z axis, it implies

$$\sum j_e = 0. \tag{11a}$$

Let us evaluate the contribution from an open diagram with two external lines j. and j'. (Fig. 3a). According to (11) and (11a) it is proportional to $\delta_{jj'} \delta_{mm'}$ where index m is contra-

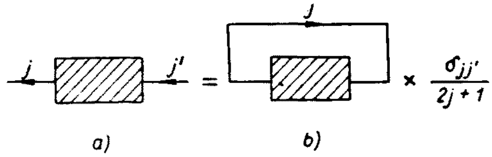


Fig. 3. Diagrams corresponding to Formula (12)

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variant and index m' covariant. Let us write this contribution as

$$(2j+1)^{-1}\, N\, \delta_{jj'}\delta_{mm'}. \tag{12}$$

If this is to hold in every coordinate system, N must be independent of m and m'. In order to evaluate it we insert in (12) $j = j'$ and contract (sum) over $m = m'$. The result is N, equal to the invariant corresponding to the diagram of Fig. 3b. Thus we get the relation shown in Fig. 3.

Going from the diagram shown in Fig. 3b to that from Fig. 3a we cut a line. Another easy operation is to remove a vertex to which an external line is attached. Let us remove the vertex shown in the diagram in Fig. 4. According to (12) the contribution of the diagram

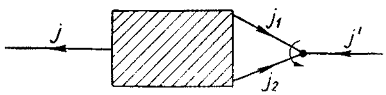

Fig. 4. Diagram which yields tensor (13) when the vertex $j_1 j_2 j'$ is removed

is

$$(j^{\cdot}\ldots)\ (\ldots j_1^{\cdot}\ldots j_2^{\cdot}\ldots)\ (j_1\cdot j_2\cdot j'\cdot) = (2j+1)^{-1}\, N\delta_{jj'}\delta_{mm'}. \tag{13}$$

Multiplying both sides by $(2j'+1)\ (j_1{}^{\cdot} j_2{}^{\cdot} j'{}^{\cdot})$, contracting over m' and summing over j' we obtain, using (8),

$$(j^{\cdot}\ldots)\ (\ldots j_1^{\cdot}\ldots j_2^{\cdot}\ldots) = N(j_1{}^{\cdot} j_2{}^{\cdot} j^{\cdot}), \tag{14}$$

which is shown on diagrams in Fig. 5. Note that the three external lines must have the same direction (all ingoing or all outgoing) because otherwise (8) would not apply. The two vertices on the right-hand side of Fig. 5 have the same orientation. As was mentioned in the preceding section only their relative orientation is relevant.

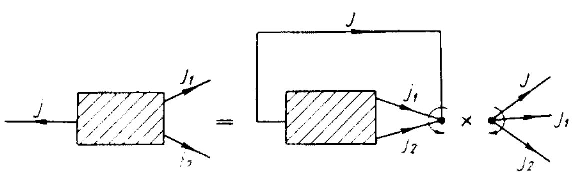

Fig. 5. Diagrams corresponding to formula (14)

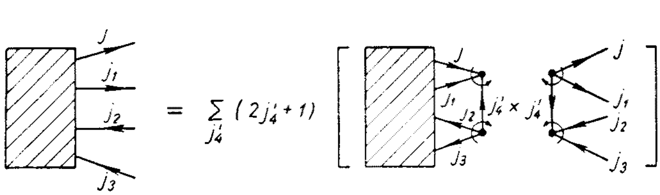

Fig. 6. Example of a more complicated identity

Removing another vertex in the same way we obtain the relation shown in Fig. 6. It is seen that this procedure yields expressions for any properly oriented open diagram which can be obtained from a closed one by removing a set of vertices adjacent to each other.

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We formulate the general theorem, which can be proved by induction. Let us denote the numbers of outgoing and ingoing lines by $n^{\cdot}$ and $n_{\cdot}$. An open diagram with $n^{\cdot}+n_{\cdot}$ even and $n_{\cdot} = n^{\cdot}$, or with $n^{\cdot}+n_{\cdot}$ odd and $n^{\cdot} = n_{\cdot} \pm 3$ reduces to a sum of products. Each product contains as factors an invariant and a tensor. Moreover it may contain factors $(2j+1)$. A detailed prescription for finding these elements is given below.

In order to draw the diagram corresponding to the invariant we start with the open diagram to be reduced. Choosing two external lines with the same orientation, we couple them by a vertex and draw from the vertex a new line (unless $n_{\cdot}+n^{\cdot} = 3$, *cf.* Fig. 5) j'_1 oriented so that at the new vertex all the lines are ingoing or all the lines are outgoing. Further we call such vertices normal. This procedure is continued until the external lines are exhausted. The last new line j'_k is coupled at the last new vertex with the last two external lines. All the new vertices must be normal.

The diagram corresponding to the tensor is composed from $k+1$ vertices coupled in the same way as the new vertices in the closed diagram described above, except that the orientations of all the lines are changed. Consequently all the vertices are normal and there are $n^{\cdot}$ outgoing lines and $n_{\cdot}$ ingoing lines. The orientation around each vertex in the closed diagram is the same as the orientation around the same vertex in an open diagram. Since the open diagram which is being reduced contains only old vertices and the open diagram from the product only new ones, this prescription is unambiguous.

The product of the invariant and the tensor should be multiplied by $(2j'_1+1)\dots(2j'_k+1)$ and summed over all the new lines j'_i. Examples are given in Figs 5 and 6. Let us note that for $n^{\cdot}+n_{\cdot}>4$ the invariant is not determined uniquely by our prescription, therefore an open diagram can be reduced in various ways.

4. *Reduction theorems*

Each reduction formula for an open diagram implies a reduction theorem for invariants which can be obtained by closing it. Let us consider the invariant corresponding to the diagram in Fig. 7a. It is equal to the contracted product of the two tensors shown in Fig. 7b. Since these give contributions of type (13) we get for the invariant

$$N = \sum_{m,m'} (2j+1)^{-1} N_A \delta_{jj'} \delta_{mm'} (2j+1)^{-1} N_B \delta_{jj'} \delta_{mm'} = (2j+1)^{-1} N_A N_B \delta_{jj'} \tag{15}$$

where N_A and N_B correspond to diagrams shown in Fig. 7c. We conclude that if a diagram can be divided into two separate parts, A and B, by cutting two lines, its invariant reduces

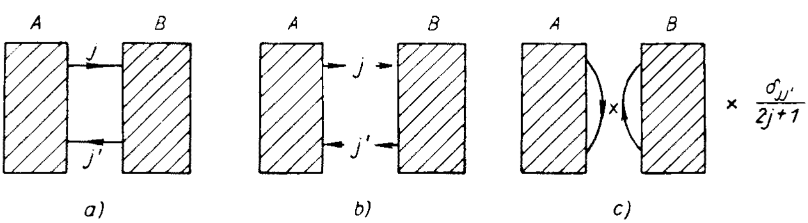

Fig. 7. Diagrams corresponding to Formula (15)

to a product of simpler invariants corresponding one to A and one to B. Before applying formula (15) the two lines to be cut should be suitably oriented. One of them must go from A to B, and the other from B to A.

Example. Carrying out the reduction shown in Fig. 8 and using Table I we obtain

$$(j_1^{\cdot} j_2^{\cdot} j_{12}^{\cdot})\ (j_{1\cdot} j_{2\cdot} j_{34\cdot})\ (\ldots j_{12\cdot} \ldots j_{34}^{\cdot}) = \delta_{j_{12}j_{34}} N(2j_{12}+1)^{-1}, \tag{16}$$

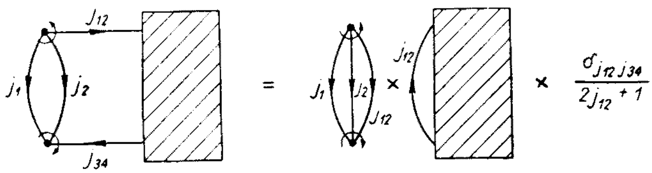

Fig. 8. Diagrams corresponding to Formula (16)

where N is the invariant obtained from the expression in the bracket by replacing in it $j_{12\cdot}$ by $j_{34\cdot}$ and contracting over m_{34}.

Similarly Formula (14) implies the reduction theorem shown in Fig. 9. Applying it, for example, to the diagram shown in Fig. 10 we obtain

$$(j_{1\cdot} j_5^{\cdot} j_{6\cdot})\ (j_{2\cdot} j_6^{\cdot} j_{4\cdot})\ (j_{3\cdot} j_4^{\cdot} j_{5\cdot})\ (j_1^{\cdot} j_8^{\cdot} j_{9\cdot})\ (j_2^{\cdot} j_9^{\cdot} j_{7\cdot})\ (j_3^{\cdot} j_7^{\cdot} j_{8\cdot}) =$$
$$= \begin{Bmatrix} j_1 j_2 j_3 \\ j_4 j_5 j_6 \end{Bmatrix} \begin{Bmatrix} j_1 j_2 j_3 \\ j_7 j_8 j_9 \end{Bmatrix}. \tag{17}$$

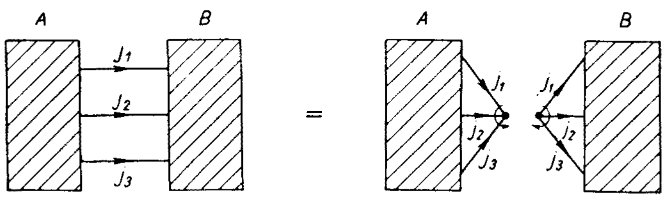

Fig. 9. Reduction used to derive Formula (17)

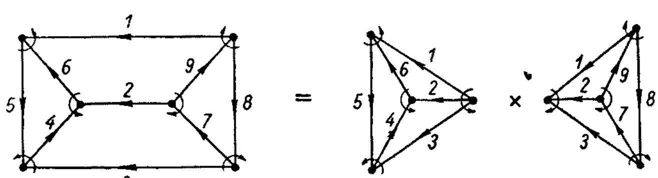

Fig. 10. Diagrams corresponding to Formula (17)

In general, whenever a diagram can be divided into two parts by cutting three lines, the invariant reduces to a product of invariants corresponding to the parts. If each of the parts contains more than one vertex, each of the factors contains less j's than the original invariant, which in this sense is reducible. Note that before cutting the lines one must orient them parallel to each other.

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It is easily seen that the only invariants which cannot be reduced to products of simpler factors and which contain not more than 12 j's are those listed in Table I.

Analogous considerations apply to a diagram which can be divided into two separate parts. A and B, by cutting $n>3$ lines. The lines to be cut must be oriented so that the theorem from the last section is applicable to part A. Reducing part A according to the general theorem, we add $n-3$ new lines. Consequently there are $n-3$ summations. The tensor obtained from the reduction must be contracted with B.

Examples.

1. Cutting the diagram 12-j (b) vertically through the centre, we obtain the product of diagrams with one summation. These diagrams, after changing the directions of some lines and around some vertices, are identical with the diagram in Fig. 10. Reducing further according to (17) we obtain finally, after a cancellation of phase factors,

$$12\text{-j (b)} = \sum_x (2x+1) \begin{Bmatrix} e j x \\ c k d \end{Bmatrix} \begin{Bmatrix} e j x \\ g l f \end{Bmatrix} \begin{Bmatrix} x i a \\ b k c \end{Bmatrix} \begin{Bmatrix} x i a \\ h l g \end{Bmatrix}. \tag{18}$$

This is the formula of Elliott and Flowers (1955). We used this formula to get the phase of our diagram with respect to the 12-j symbol.

2. Changing the orientation of lines c and g and then cutting, we obtain

$$12\text{-}j\text{ (b)} = \sum_x (2x+1) \begin{Bmatrix} a b k \\ h i g \\ l c x \end{Bmatrix} \begin{Bmatrix} x g k \\ c j d \\ l f e \end{Bmatrix}. \tag{19}$$

3. Cutting diagram 12-j (a) in the same way, we obtain the identity given by Jahn and Hope (1954)

$$12\text{-}j\text{ (a)} = \sum_x (-1)^{2x+k+g+l+c} (2x+1) \begin{Bmatrix} a b k \\ h i g \\ l c k \end{Bmatrix} \begin{Bmatrix} e j x \\ c l d \end{Bmatrix} \begin{Bmatrix} e j x \\ g k f \end{Bmatrix}. \tag{20}$$

which we used to determine the phase factor for the diagram.

Note: After this work had been completed a series of papers by Jucys *et al.* (1962) came to our notice, where similar results are obtained.

Note added in proof: We wish to thank Professor Jucys for sending us a copy of his monograph (Jucys *et al.* 1960). The method described there is practically identical with ours.

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