UNIFIED FIELD THEORY AND MODERN PHYSICS

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Five-dimensional unified field theory with a closed fifth dimension provides us with a geometrical interpretation of the electromagnetic field and of the electric charge. Gauge transformations mean translations and charge conjugation means inversion in the fifth dimension. The gravitational coupling constant is related to the Sommerfeld constant and to the circumference of the world-tube. The unifield theory predicts the existence of giant particles with rest masses of at least 10^{15} times the nucleon rest mass. There are indications that the five (or more)-dimensional approach may become useful for a future understanding of weak and strong interactions.

1. *Is unified theory out of date?*

The so called unified field theories are regarded nowadays, by most physicists, as out of date. The main reason which seems to justify this opinion is: unified theories attempt to unify the electromagnetic field with the gravitational field; such attempts appeared justified at times when only these two types of field were known. However, in view of the discoveries of other fields (spinor fields, meson fields) it is little hope to construct a unified theory of all these fields so that the problem of unification of two of them, picked more or less arbitrarily out of the great variety of fields, does not seem to be of great interest nowadays. The other reason favouring the opinion that unified field theories are out of date is that they are, as a rule, formulated in a very old-fasioned way (*e.g.* the gravitational and electromagnetic effects used to be discussed on hand of anachronic models of classical point particles or fluids).

This last argument is not very serious and may well be overcome by introducing a modernized exposition (introducing as sources the fields encountered in quantum field theory, *e.g.* the spinor fields). But also the main argument against the attempts to formulate a unified theory of gravito-electromagnetism is not convincing. The point is that gravitational and electromagnetic phenomena exhibit some striking analogies (*e.g.* photons and gravitons are the only bosons with a vanishing rest mass encountered in physics) which justifies the attempts to treat them from a unified point of view.

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It may be hoped that, later on, a more general theory will be developed in which, besides gravitation and electromagnetism, also the other phenomena will naturally find their place and appear quite necessary for reasons of completeness and internal consistency. Such a theory would also deserve the name „unified" in the more general sense of this word. Thus, interest in formulating unified theories must not become „out of date" as it means the ultimate aim of theoretical physics.

2. *A five-dimensional world-tube*

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From the many versions of unified theories we choose the five-dimensional version of Einstein *et al.*[1] whose basic assumptions are distinguished by elegance and simplicity. In this version of the unified theory the world is assumed to be a five-dimensional Riemannian tube, open in four „ordinary" dimensions and closed in the fifth dimension. (Greek indices run from 1 to 5, Latin indices from 1 to 4). It is assumed that the circumference of the tube is extremly small so that, for macroscopic observers whose spread in the ordinary dimensions is so large, the tube is practically infinitely thin and therefore it seems to us that we are living in a four-dimensional world.

In the absence of gravitation, electromagnetism, and other forms of matter the tube world may be assumed to be Minkowskian in the four open dimensions and of constant circumference in the fifth closed dimension. In a special system of coordinates we have

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu, \tag{1}$$

where

$$\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{44} = \eta_{55} = 1, \ \eta_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu. \tag{1'}$$

The coordinates $(x^k, x^5+2\pi nl)$ denote the same point for any integer n. Of course, it is possible to perform any coordinate transformation which is sufficiently regular and such that to any set of numbers x^μ corresponds one and only one point in the five-dimensional space. However, since one of the dimensions is distinguished topologically, such general transformations are not of great interest. On the other hand, of particular interest are general transformations within the four-dimensional subspace

$$x^{k'} = x^{k'}(x^l) \tag{2}$$

and phase transformations

$$x^{5'} = x^5 + lf(x^l). \tag{3}$$

A straightforward generalization of the theory of General Relativity would consist in assuming a generalized metric

$$ds^2 = dx^\mu \gamma_{\mu\nu} dx^\nu \tag{4}$$

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[1] Einstein, A., Bargmann, V., Bergmann, P., for detailed references see P. Bergmann, *Introduction to the theory of Relativity*, Prentice Hall, Inc. New York 1947.

in the five-dimensional tube-space, with the metric tensor components $\gamma_{\mu\nu}$ being periodic functions of the fifth coordinate. However, a theory involving fifteen independent metric tensor components would contradict experimental evidence: besides gravitation and electromagnetism it would involve a scalar field describing spinless particles. Such particles (with a vanishing rest mass) should have observable consequences which have never been detected.

In order to avoid this difficulty a supplementary condition, reducing the number of independent field quantities to fourteen, is needed. Einstein *et al.* have shown that if one requires the geodesic lines around the tube to be closed lines without discontinuity of direction then the circumference of the tube is constant and it is possible to choose the coordinate system so that $(x^k, x^5+2\pi nl)$ denotes the same point for any integer n whereas

$$\gamma_{55} = 1. \tag{5}$$

From this postulate it follows also that γ_{5k} are independent of the fifth coordinate

$$\frac{\partial \gamma_{5k}}{\partial x_5} = 0. \tag{6}$$

The above mentioned supplementary condition stating that the closed geodesic lines around the tube intersect at the angle zero is very simple and nautral, and does not spoil the elegant geometrical character of the theory.

3. *A variational principle*

Einstein *et al.* did not develop very far their formalism of tube-world. They argued that the Lagrangian is not unique since, besides the five-dimensional scalar curvature $\underset{(5)}{R}$, there are several other invariants with respect to the transformations (2). Moreover, they were hampered by the fact that the conditions (6) do not allow to derive purely differential equations from a five-dimensional variational principle

$$\delta \int \mathcal{L} \sqrt{-\gamma}\, d^{(5)} x = 0 \tag{7}$$

because, together with γ_{5k}, also their variations must be independent of the fifth coordinate, *i.e.* are not quite arbitrary and do not allow to get rid of the integration over the fifth coordinate.

The present author is not convinced by the above mentioned objections. First of all, in his opinion, the Lagrangian

$$\mathcal{L} = \underset{(5)}{R} \tag{8}$$

is unique (apart from a possible cosmological term). Indeed, if taking the assumption of a five-dimensional world seriously, we have to assume a Lagrangian which is scalar from the point of view of all permissible transformations in the five-dimensional world and not only from the point of view of general transformations in the four-dimensional subspace. On the other hand, the difficulty with the variational principle

$$\delta \int \underset{(5)}{R} \sqrt{-\gamma}\, d^{(5)} x = 0 \tag{9}$$

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may be circumvented in the following way: In consequence of the assumption of a closed world every field quantity must be a periodic function of the fifth coordinate and may be Fourier analysed as follows:

$$F(x^\mu) = \sum_n \overset{(n)}{F}(x^k)\, e^{inx^5/l}. \tag{10}$$

Owing to the supplementary condition we have

$$\gamma_{ij}(x^\mu) = \sum_n \overset{(n)}{\gamma_{ij}}(x^k)\, e^{inx^5/l}, \quad \overset{(n)*}{\gamma_{ik}} = \overset{(-n)}{\gamma_{ik}} \tag{11}$$

whereas

$$\overset{(n)}{\gamma_{5k}} = 0 \quad \text{for} \quad n = 1, 2, \dots \quad \text{and } \gamma_{55} = \overset{(0)}{\gamma_{55}} = 1 \tag{12}$$

while $\overset{(0)}{\gamma_{5k}}$ are real arbitrary functions of x^k.

We may Fourier analyse the integrand (9) by taking account of (11) together with the restrictions (12) and integrate over one period of the fifth coordinate. In this way we get a fourdimensional action integral

$$W = \int \mathcal{L}\, d^{(4)}x, \tag{13}$$

where the integrand is a function of $\overset{(n)}{\gamma_{ij}}(x^k)$ and $\overset{(0)}{\gamma_{5k}}(x^k)$ and their derivatives with respect to x^k. The variational principle consists in performing variations of (13) under arbitrary variations of $\overset{(n)}{\gamma_{kl}}(x^i)$ and $\overset{(0)}{\gamma_{5l}}(x^i)$.

4. *Ultra-gravitons*

The above considerations show that in the theory of a five-dimensional tube-space there appears an infinite set of field quantities $\overset{(n)}{\gamma_{ij}}$, $n = 0, 1, \dots$ and the field quantities $\overset{(0)}{\gamma_{5i}}$ being functions of the ordinary four coordinates x^k. In order to find out their physical meaning let us discuss first the case of weak fields

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{14}$$

where $h_{\mu\nu}$ are small. Assuming additionally [2]

$$\sum_k h_{kk} = 0, \quad \partial_k h_{k\mu} = 0, \tag{15}$$

our variational principle yields the following field equations:

$$(\eta^{kl}\partial_k\partial_l - n^2/l^2)\overset{(n)}{h_{ij}} = 0, \quad n = 0, \pm 1, \pm 2, \dots \tag{16}$$

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[2] Owing to (5) and (6) this may be written also five-dimensionally

$$\sum_\mu h_{\mu\mu} = 0, \ \partial_\nu h_{\nu\mu} = 0. \tag{15'}$$

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and

$$\eta^{kl}\partial_k\partial_l \overset{(0)}{h}_{5i} = 0. \tag{17}$$

(16) are Klein-Gordon equations for particles with rest masses

$$m_n = n/l, \quad n = 0, 1, \ldots \tag{18}$$

In this way it is seen that the 5-dimensional formalism predicts the existence of particles with spin equal two and with the mass spectrum (18). It will be shown later on that the constant l is closely related to the gravitational coupling constant and, in fact, is of the order of magnitude 10^{-29} cm. Thus, the rest masses are gigantic except in the case $n = 0$. It will be shown also that the particles belonging to the value $n = 0$ are gravitons and photons. The giant particles connected with the higher order harmonics $n = 1, 2, \ldots$ may be called ultra-gravitons[3].

The existence of ultra-gravitons does not contradict the present experimental evidence: We do not dispose of sufficient concentrations of energy to produce ultra-gravitons. Also the range of ultra-gravitational forces is so small that they cannot be detected. Nevertheless, their existence is of interest for purely theoretical reasons.

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5. *Unifield gravito-electromagnetic field*

Ultra-gravitons and their interactions being not detectable by the present experimental techniques, we shall leave them out of further considerations and retain only the terms with $n = 0$ in the expansion (11). This is equivalent to the assumption that the field quantities $\gamma_{\mu\nu}$ are independent of the fifth coordinate.

It is convenient to split the metric tensor components in the following way:

$$\overset{(0)}{\gamma}_{kl} = g_{kj} + l^2 f_k f_l, \quad \overset{(0)}{\gamma}_{5k} = -l f_k, \quad \overset{(0)}{\gamma}_{55} = 1. \tag{19}$$

With this decomposition the determinant of the five-dimensional manifold γ becomes identical with the determinant of g_{kl}

$$\gamma = g \tag{20}$$

and the five-dimensional scalar curvature becomes

$$\underset{(5)}{R} = \underset{(4)}{R} + \frac{l^2}{4} f_{kl} f^{kl}, \tag{21}$$

where $\underset{(4)}{R}$ is the usual expression for the scalar curvature of the four-dimensional subspace with the metric g_{kl} while

$$f_{kl} = \partial_l f_k - \partial_k f_l, \quad f^{kl} = g^{ki} g^{lj} f_{ij}. \tag{22}$$

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[3] Owing to (6) and (17) ultra-photons with non-vanishing rest masses do not exist.

The 5-dimensional variational principle (9) reduces to a four-dimensional one

$$\delta \int \left(\underset{(4)}{R} + \frac{l^2}{4} f_{kl} f^{kl} \right) \sqrt{-g} \, d_x^{(4)} = 0 \tag{23}$$

with g_{kl} and f_k to be varied independently. It is immediately seen that f_{kl} is interpretable as a quantity proportional to the electromagnetic field whereas l^2 is interpretable as a constant proportional to the gravitational coupling constant. The factor of proportionality may be determined only when taking into consideration the sources of the electromagnetic field.

From the variational principle (23) we get equations of the gravitational field with sources in the form of energy-momentum-stress tensor of the electromagnetic field, and Maxwell equations in the presence of the gravitational field. The unified field theory includes automatically the interaction of these fields.

As is seen from (19) the electromagnetic potentials f_k (up to a constant factor) possess a simple geometrical meaning: they are proportional to the components γ_{5k} of the metric tensor of the five-dimensional world. Gauge transformations

$$f_k \to f_k + \partial_k f \tag{24}$$

possess also a simple geometrical meaning: they are nothing else but the phase transformations (3).

6. *Phasor*

In spaces with a closed dimension there exists, besides tensors, another interesting geometrical object to be called „phasor". It is defined as a complex quantity ψ transforming under phase transformations (3) according to

$$\psi \to \psi \, e^{if}, \ \psi^* \to \psi^* e^{-if}. \tag{25}$$

The real and imaginary parts of a phasor form a vector in the two-dimensional space embedding the closed dimension. A phasor may be, at the same time, any tensor or spinor under the transformations (2).

Since the partial derivatives of a phasor are no more phasors, we have to introduce the concept of covariant differentiation of phasors. If ψ is scalar under the transformations (2) then $\psi_{;k}$ defined by

$$\psi_{;k} = (\partial_k - i f_k) \, \psi \tag{26}$$

are phasors. On the other hand $\psi_{;k} \equiv \partial_5 \psi$ is a phasor if ψ is a phasor, due to the fact that the phase f is independent of the fifth coordinate.

7. *Geometrical interpretation of electric charge and connections between fundamental constants*

A comparison of the Formula (26) with the well-known rule for introducing interaction of charged fields ψ with the electromagnetic field

$$\partial_k \psi \to (\partial_k - ie \, A_k) \, \psi, \tag{26'}$$

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shows that complex fields carrying electric charge are phasors, while the rule (26') means nothing else but transition to a covariant differentiation. Moreover, it is seen that the factor of proportionality between the electromagnetic potentials A_k and the quantities f_k involved in the metric (19) is the elementary charge

$$f_k = eA_k. \tag{27}$$

From (23) and (27) it follows that the funfamental length l (the radius of the world-tube) is connected with the gravitational coupling constant $\varkappa$ as follows:

$$\varkappa = e^2 l^2. \tag{28}$$

This settles the numerical value of l

$$l \simeq 10^{-29} \text{ cm}. \tag{28'}$$

The five-dimensional theory does not provide us with an explanation why the numerical value of the Sommerfeld constant is about 1/137 but puts this problem in a new context: why the gravitational coupling constant (in units $c = \hbar = 1$) is neither simply equal to the square of the length l^2, nor equal to πl^2, or something of this sort, but is equal to $l^2/_{137}$?

As is well-known, the operator of electric charge associated with charged fields is interpretable as the operator of infinitesimal phase transformation. In the five-dimensional theory this operator acquires a geometrical interpretation: it is the operator of infinitesimal displacement of the closed fifth coordinate. In other words, the electric charge is the momentum canonically conjugate to the coordinate x^5. Since displacement of this coordinate means also rotation in a two-dimensional space embedding the closed fifth dimension, the electric charge is also interpretable as angular momentum in this embedding space.

Gauge transformations mean translations of x^5. Besides gauge transformations also charge conjugation

$$\psi \rightleftarrows \psi^* \tag{29}$$

acquires a geometrical interpretation: it is inversion of the fifth coordinate. This is very interesting and satisfactory in view of the discovery that, although ordinary parity is not always conserved, the physical laws remain invariant under the CP-transformation. The very fact that charge conjugation makes up for what is spoiled by reflection of the ordinary (space-like) coordinates provided a hint that charge conjugation must have something to do with reflections. The five-dimensional unified field theory clarifies this relation by interpreting charge conjugation C as reflection in the fifth dimension (which is also space-like). Thus, the CP-invariance means that physical laws are invariant under simultaneous reflections of all (four) space-like coordinates.

8. Sources of the unified field

In the foregoing section we discussed a unified theory of gravitational and electromagnetic field including automatically their mutual interaction: the electromagnetic field provides sources of the gravitational field and is modified itself by the latter since the metric tensor

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components g^{ji} are involved in the Lagrangian of the electromagnetic field appearing in (22) and (23). The next problem is to introduce other sources of the unified field. Instead of dealing with anachronic „point particles" or „classical fluids" let us introduce the fields playing a role in contemporary physics. Consider *e.g.* a scalar complex field ψ described usually by means of the Lagrangian

$$\mathcal{L} = \partial_k \psi^* \ \eta^{kl} \gamma_l \psi + m^2 \psi^* \psi. \tag{30}$$

This Lagrangian has to be rewritten in a covariant form, multiplied by the gravitational coupling constant, and added to the Lagrangian $\underset{(5)}{R}$ of the unified field. A natural generalization of (30) for the case of a five-dimensional theory is

$$\mathcal{L} = \overset{*}{\psi}_{;\mu} \gamma^{\mu\nu} \psi_{;\nu} + m^2 \psi^* \psi, \tag{30'}$$

where ψ is a function of the five coordinates x^μ, periodic in the fifth coordinate. A decomposition of all quantities ψ, ψ^*, and $\gamma^{\mu\nu}$ according to (10) would yield terms involving ultraparticles with gigantic masses n/l and their interactions which practically play no role so that again, we may limit ourselves only to the terms $n = 0$ and reduce the five-dimensional Lagrangian (30') to a four-dimensional one

$$\mathcal{L} = \overset{*}{\psi}_{;k} \gamma^{kl} \psi_{;l} + m^2 \psi^* \psi. \tag{30''}$$

According to (19), (27), and (28) we have

$$\gamma_{kl} = g_{kl} + \varkappa A_k A_l, \tag{31}$$

which shows that γ_{kl} differ very little from g_{kl}. Limiting ourselves to the terms of the first order in the gravitational coupling constant we may replace in (300'') γ^{kl} by g^{kl} and are left with the variational principle

$$\delta \int \left(\underset{(4)}{R} + \frac{\varkappa}{4} F_{kl} F^{kl} + \varkappa \mathcal{L} \right) \sqrt{-g}\, d^{(4)}x = 0, \tag{32}$$

with

$$\mathcal{L} = \overset{*}{\psi}_{;k} g^{kl} \psi_{;l} + m^2 \psi^* \psi, \tag{32'}$$

where the covariant derivatives have to be understood in the sense of the covariant derivatives of a phasor (26).

The variational principle includes the usual interaction of the charged field ψ with the gravitational and the electromagnetic field.

9. *Outlook*

In the foregoing sections it was shown that the five-dimensional theory yields a geometrical interpretation of the electromagnetic field and of the electric charge. It throws also some light upon the meaning of the gravitational coupling constant and its connection with the Sommerfeld constant and the radius of the world-tube. Undoubtedly, this version of

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unified field theory is principally by far more satisfactory than the usual non-unified theory. It seems also to offer new hopes for the future. First of all, the interpretation of charge conjugation as inversion of the fifth coordinate may contribute to a future better understanding of the problem of week interactions with their characteristic property of violating the conservation of parity. Moreover, the five-dimensional theory seems also to throw some light upon the puzzle of isospin and to provide a starting point for its future explanation.

Two facts seem to us very remarkable: (I) There exists a connection between the electric charge and the third component of isospin, *e. g.* $Q = I_3 + b/2$. (II) The electric charge acquired a geometrical meaning in the 5-dimensional space in terms of the operator of infinitesimal displacement in the fifth dimension (or of infinitesimal rotation in a space embedding the fifth dimension). From these two facts it follows that also the third component of isospin is interpretable geometrically in terms of rotation in the same embedding space. Consequently, not only the third component but any component of isospin must possess a similar geometrical meaning. In other words, there should exist an isospace with the same amount of physical reality as our ordinary space, while one of the dimensions of isospace must be identical with the fifth dimension of the world-tube.

It is possible that this common (fifth) dimension constitutes the only connection between the two spaces. It is plausible that (similarly as is the fifth dimension) also the other dimensions of isospace are closed so that all operators of infinitesimal displacements in isospace possess discrete spectra of eigenvalues and can be used for classification of strongly interacting particles. A closer investigation of the characteristic properties of baryons and mesons, in particular of the symmetries of their interactions, may supply us with a detailed information about the dimensionality and topological as well as metrical properties of isospace. Thus, some hope seems to be justified for a future development of the unified theory involving the phenomena of strong interactions.

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