THE "VECTORIAL" OPTICS OF FIELDS WITH ARBITRARY SPIN, REST-MASS ZERO

By Jerzy Plebański*

(*Received June 7, 1964*)

The equations of fields with spin s, rest-mass zero, corresponding to the irreducible representation of the Lorentz group $D(o, s)$ are studied in the high frequency limit. A formula for the rotation of the planes of polarization-due to curvature — is given. The optical development, *i.e.*, the development of the field with extracted phase factor into inverse powers of frequency, exhibits in its algebraic structure a characteristic „peeling off" effect.

1. *Introduction*

The standard "scalar optics" based on the eiconal equation studies the properties of light rays (*i.e.* bi-characteristics). The "vectorial" structure of the field under consideration (amplitude, polarization, *ect.*) lies almost entirely outside of the scope of the "scalar optics".

The methodics of the optical approximation (*i.e.*, physically, of the high frequency approximation) can be so adopted that it yields an appreciable amount of information not only about bi-characteristics but also about the "vectorial" structure of the studied field.

In order to develop "vectorial optics" one does not need to enter into the fine details of the diffraction theory. It is enough to study the integrability conditions of eqations determining higher terms of the "optical development". (This development is the development of the field with extracted phase factor into inverse powers of the frequency). The integrability conditions impose some relevant restrictions on the terms of lower order. The fundamental idea of this trick — as applyied to the electromagnetic field scattered by week gravitation — has been already used in [1], [2], but in a non-covariant manner.

The aim of this paper is to develop in a fully covariant formulation the "vectorial optics" of fields with the arbitrary spin s but rest-mass zero. The cases of particular interest are of course: 1) $s = \frac{1}{2}$, the neutrino field 2) $s = 1$, the electromagnetic field 3) $s = 2$ "gravitational radiation".

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* Address: El Centro de Estudios Avanzados del Instituto Politécnico Nacional, México, Mexico City. On the leave of absence from the University of Warsaw, Poland.

We will work in a general non-flat space-time, a normal hyperbolic Riemannian space V_4 with the signature $(+---)$. By doing so we will be able to approach the problems: (1) of the behaviour of "optical" beams of neutrinos as scattered by gravitational field; (2) the same for electromagnetic waves[1]; (3) the optics of the "gravitational radiation". The last because of the very spirit of "gravitational radiation" has to be studied on the level of the Riemannian geometry.

Naturally, all results derived in V_4 apply *mutatis mutandi* when the curvature (*i.e.*, gravitation) can be neglected and V_4 becames a flat pseudo-Euclidean space S_4. Nevertheless, the general covariance inherited from V_4 is also useful in S_4. The arbitrarnes of curvilinear coordinates allows one to discuss waves of different symmetries (*e.g.*, plane, spherical, *etc.*) on an equal footing.

The eqations studied in this paper have the same form when the physical fields are understood as "test-fields" not affecting the space-time or when understood as fields which generate gravitation through theirs energy-momentum tensors. Therefore, most of results obtained apply to both cases. The distinguished case of $s = 2$, where the physical field consists from the conformal curvature itself, must to be dealt with special care.

Of course, one can talk about the spin 2 waves in S_4 or on the background of V_4. But such waves as different from the conformal curvature have little in common with the genuine "gravitational radiation".

In Section II the fundamental chain of recurrence relations following from the "optical development" will be given. Section III will provide us with a resume of the theory of optical congruences as adopted to our purposes. In Section IV the potential congruences will be studied. Emphasis will be concentrated upon some differential identities which establish relationships between the derivatives of optical scalars (also quasi-scalars) and the curvature. Section V will introduce a convenient basis in the space of the spinors $\psi_{A_1 \ldots A_{2s}} = \psi_{(A_1 \ldots A_{2s})}$. With the help of this basis equations of the "vectorial optics" will be reduced in Section VI to a set of eqations for some "scalar amplitudes". The last equations in low orders are investigated in Section VII.

Section VIII offers a "spinorial" theory of the planes of polarization. Finally, Section IX will give the physical interpretation of the results derived on the level of "vectorial optics".

II. The statment of the problem

Let $\nabla^{\dot{A}B} = g^{\alpha\dot{A}B}\, \nabla_\alpha$ denote the operator of spinorial covariant differentiation in V_4. The fields with spin s and vanishing rest-mass which correspond to the irreducible representation of the Lorentz group $D(s, o)$ (in tangential space) are the spinorial fields

$$\psi_{A_1 \ldots A_{2s}} = \psi_{(A_1 \ldots A_{2s})}, \qquad s = \frac{1}{2}, \quad 1, \frac{3}{2}, \ldots, \tag{2.1}$$

[1] The papers [1], [2] studied this last problem but in the approximation of the week gravitational field and in not fully covariant manner.

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submitted to the dynamical equations

$$\nabla^{\dot{A}B_1} \psi_{B_1 B_2 \dots B_{2s}} = 0. \tag{2.2}$$

The are other possible irreducible representations corresponding to spin s and related to these field equations with the rest-mass zero; see [3], [4], [5], [6]. We restrict ourselves in this paper to the field equations (2.2) as Penrose does in his last works [7], [8] — because they cover the most familiar physical cases: $s = \frac{1}{2}$, the 2 component neutrinos, $s = 1$, the Maxwell's equations without currents in spinorial formulation. For $s = 2$, (2.2) may be understood as the Bianchi identities in Einsteinian space ($R_{\alpha\beta} \equiv 0$) expressed in terms of the spin image of the conformal curvature: $W_{\alpha\beta\gamma\delta} \leftrightarrow W_{ABCD} \equiv \psi_{ABCD}$, [9]. The field ψ_{ABCD} can be interpreted as the field of "pure" gravitational radiation; the spin image of the Bell—Robinson tensor [10], [11] related to it is simply

$$\psi_{\dot{A}_1 \dots \dot{A}_4} \, \psi_{B_1 \dots B_4}.$$

We are interested in solutions of (2.2) of the form:

$$\psi_{A_1 \dots A_{2s}} = \sum_{n=0}^{\infty} (ik)^{-n} \, \psi^{(n)}{}_{A_1 \dots A_{2s}} \exp\,[ik\Phi], \tag{2.3}$$

where k is very large and the "partial waves" $\psi^{(n)}_{A_1 \dots A_{2s}}$ are also symmetric in all indices (the "optical development"). The k may be understand as related to the dominant frequency of the field.

The fundamental idea of the "optical development" is very simple: the phase factor $\exp[ik\Phi]$ is supposed to provide us with the dominant variation of the field, its gradient is very large. The amplitude is supposed to have a limit when $k \to \infty$; therefore it is reasonable to expect it to be a power series in l/ik.

Substituting (2.3) into (2.2), cancelling the factor $\exp[ik\Phi]$ and equating the coefficients at different powers of ik to zero, we obtian the following chain of equations for the succesive approximations:

$$(\nabla^{\dot{A}B_1}\Phi) \cdot \psi^{(n+1)}_{B_1 \dots B_{2s}} + \nabla^{\dot{A}B_1} \psi^{(n)}_{B_1 \dots B_{2s}} = 0, \tag{2.4}$$

$$n = -1,\ 0,\ 1,\ 2,\ \dots,$$

where, for conveniency, we introduced $\psi^{(-1)}_{A_1 \dots A_{2s}} \equiv 0$.

We should like to integrate these equations step by step. Setting $n = 1$ in (2.4) we get

$$(\nabla^{\dot{A}B_1}\Phi) \cdot \psi^{(0)}_{B_1 \dots B_{2s}} = 0. \tag{2.5}$$

We must however assume that

$$\Delta^{\dot{A}B}\Phi \neq 0, \quad \psi^{(0)}_{A_1 \dots A_{2s}} \neq 0, \tag{2.6}$$

i.e., that Φ has not-trivial gradient and that the optical development has a not-trivial beginning. Suppose now that Det $||\nabla^{\dot{A}B}\Phi|| \neq 0$. If so, there exists a matrix $||M_{C\dot{A}}||$ such that $M_{C\dot{A}}\nabla^{\dot{A}B}\Phi = \delta^B_C$. Contracting (2.5) with $M_{C\dot{A}}$ we would get $\psi^{(0)}_{A_1 \dots A_{2s}} = 0$ which contradicts (2.6).

Therefore, one has

$$\mathrm{Det}||\nabla^{\dot{A}B}\Phi|| \equiv \tfrac{1}{2}(\nabla_{\dot{A}B}\Phi)\cdot(\nabla^{\dot{A}B}\Phi) \equiv \Phi^{;\alpha}\Phi_{;\alpha} = 0. \tag{2.7}$$

Hence, as it was to be expected, the phase Φ must obey the eiconal equation.

The study of the chain of eqations (2.4) splits into two parts:

1) The study of "scalar" optics based on (2.7), *i.e.*, the study of the potential congruences of "light rays" (in inverted comas because there may be either neutrinos-rays or gravitational rays, *etc.*, for different values of s).

2) The study of the chain of equations (2.4) when the properties of $\nabla^{\dot{A}B}\Phi$ are already known from the "scalar" optics, *i.e.*, what we understand by "vectorial" optics.

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III. Congruences of light rays — general properties

The study of the congruences of light rays in a few past years stimulated important developments in general relativity [12], [13], [14], [15], *etc.* The theory of congruences forms actually a known, well developped useful tool in the field of all problems related to the algebraic of the curvature. This section does not add anything essentially new to the theory of congruences; its aim is nearly to adopt some elements of this theory to our purposes and to provide us with the clear geometric interpretation which we will need on the level of "vectorial optics".

The relations

$$x^\alpha = X^\alpha(\tau, p^1, p^2, p^3), \tag{3.1a}$$

$$\mathrm{J} \underset{\mathrm{df}}{=} \frac{\partial(x^0x^1x^2x^3)}{\partial(\tau p^1p^2p^3)} \neq 0 \tag{3.1b}$$

represent a congruency of light rays when

$$t^\alpha \underset{\mathrm{df}}{=} \frac{\partial X^\alpha}{\partial\tau} \qquad t^0>0, \qquad t^\alpha t_\alpha = 0, \qquad \frac{Dt^\alpha}{\partial\tau} = 0 \tag{3.2}$$

($t^0>0$ means that we restrict ourselves to the light rays "moving" into the future with increasing affine parameter τ). Because of (3.1b), relations (3.1a) are r e v e r s i b l e and, therefore, t^α may be understood as $t^\alpha = t^\alpha(x)$, *i.e.*, as a f i e l d in V_4. Obviously, the light rays (3.1a) are the integrals of $\frac{dx^\alpha}{d\tau} = t^\alpha[x(\tau)]$. The field $t^\alpha(x)$ obeys the algebraic and differential conditions

$$t^0>0, \qquad t^\alpha t_\alpha = 0, \tag{3.3a}$$

$$t^\beta t^\alpha{}_{;\beta} = 0. \tag{3.3b}$$

Now, we would to give the geometric interpretation of the matrix of derivatives of t^α: $M \underset{\mathrm{df}}{=} ||t^\alpha{}_{;\beta}||$.

Consider a pair of events x^α, $x^\alpha+\delta x^\alpha$ linked by an infinitesimal vector δx^α. When the affine parameter τ increases by $d\tau$ our events "shift" along the corresponding light rays to the positions $x^\alpha+t^\alpha(x)d\tau$, $x^\alpha+\delta x^\alpha+t^\alpha(x+\delta x)d\tau$. The vector which links these, $\delta x^\alpha+$ $+t^\alpha{}_{,\beta}\delta x^\beta d\tau$, after being parallely displaced from $x^\alpha+t^\alpha(x)d\tau$ to x^α (along $-t^\alpha(x)d\tau$) may be symbolically represented as

$$\delta x' = (1+Md\tau)\delta x \tag{3.4}$$

(1 stands here for the unit matrix). This formula may be understood as an infinitesimal transformation of δx. This transformation — in the analogy with the standard procedure of the mechanics of the continuous media can be thought of as the product of three more elementary infinitesimal transformations (and not dependent on their order). These are: 1) an infinitesimal Lorentz rotation; 2) a "volume" preserving deformation; 3) a shape preserving deformation (where $\delta x'$ is just proportional to δx, *i.e.*, a "magnification"). The only peculiarities encountered here are the 4-dimensional formulation and the fact obvious form (3.3b) that (3.4) leaves unchanged any vector proportional to the null vector t_α.

Now, decompose M as follows

$$M = M_1+M_2; \quad M_1 \underset{\mathrm{df}}{=} ||t^\alpha_{(\cdot\,;\,\beta)}||, \quad M_2 = ||t^\alpha_{[\cdot\,;\,\beta]}||. \tag{3.5}$$

The "skew" M_2, of course, generates an infinitesimal Lorentz transformation.

It follows from (3.3a—b) that $t_{[\alpha;\beta]}t^\beta = 0$. This implies 1) that transformation M_2 preserves t^α; *i.e.*, $(1+M_2d)t = t$, and 2) that $\mathrm{Det}||t_{[\alpha;\beta]}|| = 0$, so that the bi-vector $t_{[\alpha;\beta]}$ is simple. There are two distinct algebraic possibilities: the bi-vector $t_{[\alpha;\beta]}$ may be either null or general. It presents no difficulty to find the canonical form of $t_{[\alpha;\beta]}$ in both cases.

The easiest method of finding these canonical forms consists in the application of the null-leg formalism. We will need this formalism also for other purposes; in fact, it will be crucial in further sections. Therefore, we sketch it here.

The null vector t^α (with $t^0>0$) may be always represented

$$t_\alpha = \frac{1}{2}\, g_{\alpha\dot{A}B}t^{\dot{A}}t^B, \tag{3.6}$$

the spinor t_A being determined through t_α with accuracy up to

$$t_A \to t'_A = t_A e^{i\theta}. \tag{3.7}$$

One picks up a spinor s_A linearly independent to it and normalized by

$$t^A s_A = 1. \tag{3.8}$$

This spinor is determined with accuracy up to

$$s_A \to s'_A = s_A e^{-i\theta}+t_A\beta, \quad \beta \text{ complex}, \tag{3.9}$$

where Θ is the same as in (3.7).

From the spinors t_A, s_A one constructs a null-leg (sachs-leg):

$$t_\alpha = \frac{1}{2}\, g_{\alpha\dot{A}B}t^{\dot{A}}t^B, \quad s_\alpha = \frac{1}{2}\, g_{\alpha\dot{A}B}s^{\dot{A}}s^B,$$

$$m_\alpha = \frac{1}{2}\, g_{\alpha\dot{A}B}t^{\dot{A}}s^B, \quad \overline{m}_{\dot{\alpha}} = \frac{1}{2}\, g_{\alpha\dot{A}B}s^{\dot{A}}t^B. \tag{3.10}$$

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The only scalar products $\neq 0$ between the four null vectors t_α, s_α, m_α, $\overline{m}_\alpha$ are:

$$t^\alpha s_\alpha = 1/2, \qquad m^\alpha \overline{m}_\alpha = -1/2. \tag{3.11}$$

With the help of 4 null vectors one constructs an orthonormal, rightly oriented four-leg (tetrad) as

$$g^{\hat{0}}_\alpha = t_\alpha = s_\alpha, \; g^{\hat{3}}_\alpha = t_\alpha - s_\alpha, \tag{3.12}$$

$$g^{\hat{1}}_\alpha = m_\alpha + \overline{m}_\alpha, \quad g^{\hat{2}}_\alpha = \frac{1}{i}(m_\alpha - \overline{m}_\alpha),$$

with the properties:

$$g_{\hat{\alpha}\hat{\beta}} g^{\hat{\alpha}}_\alpha g^{\hat{\beta}}_\beta = g_{\alpha\beta}, \quad g^{\hat{0}}_0 > 0, \quad \mathrm{Det}\|g^{\hat{\alpha}}_\alpha\| = \sqrt{-g} > 0. \tag{3.13}$$

($\hat{\alpha}$ — the tetradial index-just labels different vectors of the four-leg and has nothing to do with vectorial indices $\alpha, \beta, \ldots$ It runs through: $\hat{\alpha} = \hat{0}, \hat{1}, \hat{2}, \hat{3}$; by $g_{\hat{\alpha}\hat{\beta}}$ we understand

$$\|g_{\hat{\alpha}\hat{\beta}}\| \underset{\mathrm{df}}{=} \|\mathrm{Diag}\,(1, -1, -1, -1)\|.)$$

When the rightly oriented tetrad $g^{\hat{\alpha}}_\alpha$ is given, the elementary spinors t_A and s_A which generate it are determined with accuracy up to a simultaneus change of signs.

Now, in the case of $t_{[\alpha;\beta]}$ with the property $t_{[\alpha;\beta]} t^\beta = 0$ the Θ and β in (3.7), (3.9) may be chosen so that $t_{[\alpha;\beta]}$ can be represented in one of the two alternative forms:

$$t_{[\alpha;\beta]} = -2\varrho g^{\hat{1}}_{[\alpha} g^{\hat{2}}_{\beta]} \quad \text{(general case)} \tag{3.13a}$$

$$t_{[\alpha;\beta]} = -2\varrho' t_{[\alpha} g^{\hat{2}}_{\beta]} \quad \text{(null case)} \tag{3.13b}$$

Of course $\varrho^2 = \frac{1}{2} t_{[\alpha;\beta]} t^{[\alpha;\beta]}$; in the null case ϱ^2 vanishes. In the general case the plane of $g^{\hat{1}}$ and $g^{\hat{2}}$ is invariantly distinguished. One easily sees that the vector $\delta x_\alpha = \alpha x^1 g^{\hat{1}}_\alpha + \delta x^2 g^{\hat{2}}_\alpha$ which lies in this plane, under the infinitesimal transformation $1 + M_2 d\tau$ performs an infinitesimal rotation through the angle $d\vartheta = \varrho d\tau$ (in direction from $g^{\hat{1}}$ to $g^{\hat{2}}$, "around" the propagation vector t_α); this is the rotation of the optical image. Hence:

$$\varrho^2 = \left(\frac{d\vartheta}{d\tau}\right)^2 = \frac{1}{2} t_{[\alpha;\beta]} t^{[\alpha;\beta]}. \tag{3.14}$$

The quantity ϱ, the first optical scalar, is called "rotation".

In the null case of (3.13b) a similar but not so obvious interpretation may be attached to the quasi-scalar ϱ'. We will not elaborate on this case.

Consider now the "symmetric" $M_1 = \|t^\alpha_{(\cdot;\beta)}\|$. Also equations (3.3a—b) here imply $t_{(\alpha;\beta)} t^\beta = 0$. Consequently, one has 1) $\mathrm{Det}\,\|t_{(\alpha;\beta)}\| = 0$ and 2) the transformation $1 + M_1\, d\tau$ preserves the vector t^α.

By applying the formalism of (3.6—3.13) one can prove by appropriate choice of Θ, β that one can always construct such a tetrad that $t_{(\alpha;\beta)}$ may be represented in one of

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the two possible canonical forms in terms of these tetrads:

$$t_{(\alpha;\beta)} = \xi t_\alpha t_\beta - \frac{1}{2}(\Theta+\sigma) g^{\hat{1}}_\alpha g^{\hat{1}}_\beta - \frac{1}{2}(\Theta-\sigma) g^{\hat{2}}_\alpha g^{\hat{2}}_\beta, \tag{3.15a}$$

(when $\Theta^2-\sigma^2 \neq 0$).

$$t_{(\alpha;\beta)} = -2\eta t_{(\alpha} g^{\hat{2}}_{\beta)} - \Theta g^{\hat{1}}_\alpha g^{\hat{2}}_\beta, \tag{3.15b}$$

(when $\Theta^2-\sigma^2 = 0$).

The optical scalars Θ and σ are the invariants:

$$\Theta \underset{\mathrm{df}}{=} t^\alpha{}_{;\alpha} \text{ (the “divergence” or “magnification”)} \tag{3.16a}$$

$$\sigma \underset{\mathrm{df}}{=} [2t_{(\alpha;\beta)} t^{(\alpha;\beta)} - t^\alpha{}_{;\alpha} t^\beta{}_{;\beta}]^{1/2} \text{ (the “sheer”).} \tag{3.16b}$$

The quasiscalars ξ and η are of secondary importance; their importance lies in whether ξ or η are zero or non-zero as that influences the order of the minimal polinomial of the matrix $M_1 = ||t^\alpha{}_{(\cdot;\beta)}||$ (the polinomial $P(\lambda) = 1.\lambda^N + a\lambda^{N-1} + \ldots$ with the property $P(M_1) = 0$, of the lowest possible order N).

In this paper we will concentrate on $t_{(\alpha;\beta)}$ of the general form (3.15a), leaving the detailed investigation of the more pathological $t_{(\alpha;\beta)}$, of the form (3.15b) for the future.

In the general case of (3.15a) when $\Theta \neq 0$, $\sigma \neq 0$, $\sigma^2-\Theta^2 \neq 0$ the tetrad is uniquely fixed, so that the $g^{\hat{1}}$, $g^{\hat{2}}$-plane is invariantly distinguished. It is easy to see the interpretation of Θ and σ in this case. Under the infinitesimal transformation $1+M_1 d\tau$, the vector $\delta x_1 g^{\hat{1}}_\alpha + \delta x_2 g^{\hat{2}}_\alpha$ from that plane changes into

$$\left[1+\frac{1}{2}(\Theta+\sigma)\, d\tau\right] \delta x_1 g^{\hat{1}}_\alpha + \left[1+\frac{1}{2}(\Theta-\sigma)\, d\tau\right] \delta x_2 g^{\hat{2}}_\alpha.$$

It follows that, under the transformation related to Θ, any area in the $g^{\hat{1}}_\alpha$, $g^{\hat{2}}_\alpha$-plane changes from δS into $\delta S + d\delta S = (1+\Theta d\tau)\delta S$ without any change of its shape. Consequently

$$\Theta = \frac{1}{\delta S}\frac{d\delta S}{d\tau}, \tag{3.17}$$

which justifyies the name “magnification”. Under the infinitesimal transformation related to σ a geometrical figure in the $g^{\hat{1}}$, $g^{\hat{2}}$-plane preserves the area but changes its shape: $\delta x_1' = (1+\frac{1}{2}\sigma d\tau)\delta x_1$, $\delta x_2' = (1-\frac{1}{2}\sigma d\tau)\delta x_2$; a cricle of the radius δr becomes an ellipse with axes $\delta a = (1+\frac{1}{2}\sigma d\tau)\,\delta r$ and $\delta b = (1-\frac{1}{2}\sigma d\tau)\delta r$ and with the square of excentricity $\varepsilon^2 = 1-(\delta b/\delta a)^2 = 2\sigma d\tau$. These facts give a clea rgeometric interpretation of σ and justify its name “shear”.

A similar interpretation may be given to $\sigma = |\Theta|$ in the case of the canonical form (3.15b) with some modifications due to the presence of the term proportional to η.

The directional derivatives of ϱ, Θ, σ “along light rays”, $\nabla_t = t^\alpha \nabla_\alpha$ can be easily computed as

$$\nabla_t \varrho + \varrho\Theta = 0, \tag{3.18}$$

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$$\nabla_t \Theta + \frac{1}{2} [\Theta^2 + \sigma^2 - (2\varrho)^2] = -\frac{1}{2} U_{\dot{A}\dot{B}CD} t^{\dot{A}} t^{\dot{B}} t^C t^D$$

$$\sigma \nabla_t \sigma + \sigma^2 \Theta + \Theta (2\varrho)^2 = 4\sigma \operatorname{Re} (W_{ABCD} t^A t^B t^C t^D).$$

The spinor t_A entering here is that which induces the tetrad corresponding to the canonical form of $t_{(\alpha;\beta)}$, (3.15a). The curvature spinors W_{ABCD}, $U_{\dot{A}\dot{B}CD}$ entering in these formulae are the spinor images of Weyl's tensor of the conformal curvature and of the Ricci tensor with extracted trace:

$$W_{\alpha\beta\gamma\delta} = S_{\alpha\beta}{}^{AB} W_{ABCD} S_{\gamma\delta}{}^{CD} + C.C. \tag{3.19}$$

$$R_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R = \frac{1}{2} g_{\alpha\dot{A}C} g_{\beta\dot{B}D} U^{\dot{A}\dot{B}CD},$$

(the spin-tensor $S_{\alpha\beta AB}$ is defined through the Pauli matrices $g_{\alpha\dot{A}B}$ *as* $S_{\alpha\beta AB} = g_{\dot{R}\dot{S}} g^{\dot{R}}_{\alpha(A} g^{\dot{S}}_{|\beta|B)}$).

IV. Potential congruences differential identities

The congruency is said to be potential when $t_{[\alpha;\beta]} \equiv 0$ which implies $t_\alpha = \Phi_{;\alpha}$ is the eiconal function; of course here $t_{(\alpha;\beta)} = \Phi_{;\alpha\beta}$ and the rotation ϱ identically vanishes. Taking these facts into account one can freely apply all formulae of the previous section, in particular the canonical forms of $t_{(\alpha;\beta)} = \Phi_{;\alpha\beta}$, *i.e.*, (3.15a—b). When $\sigma \neq 0$, $\Theta^2 - \sigma^2 \neq 0$ (the canonical form (3.15a)) or when $\sigma = |\Theta| \neq 0$, $\eta \neq 0$ (the canonical form (3.15b)) the spinors generating the tetrad to which the canonical form is referred are determined with accuracy up to a simultaneus change of sign. Therefore, one has the formulae

$$t_{A;\alpha} = \omega_\alpha t_A + \nu_\alpha s_A \tag{4.1}$$

$$\omega_\alpha = \frac{1}{2} (\xi_\alpha + i\eta_\alpha), \; \xi_\alpha, \; \eta_\alpha \text{ real,}$$

$$s_{A;\alpha} = -\omega_\alpha s_A + \mu_\alpha t_A$$

which determine three geometrically determined complex vectors ω_α, ν_α, and μ_α. The coefficients of t_A in $t_{A;\alpha}$ and of s_A in $s_{A;\alpha}$ have to be the same but with opposite sign to guarantee consistency with the normalization condition $t^A s_A = 1$.

The chief aim of this section is to investigate the properties of these vectors under the assumption that t_A and s_A are rigidly fixed by the canonical form of $\Phi_{;\alpha\beta}$. A geometric interpretation of these vectors will be also given. On the level of the "vectorial" optics these three vectors are of crucial importance.

The ξ_α and ν_α are the simplest of the vectors considered. Indeed, applying (4.1) one obtains

$$\Phi_{;\alpha\beta} = t_{\alpha;\beta} = \frac{1}{2} g_{\alpha\dot{A}B} (t^{\dot{A}} t^B)_{;\beta} = t_\alpha \xi_\beta + m_\alpha \nu_\beta + \overline{m}_\alpha \overline{\nu}_\beta. \tag{4.2}$$

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On the other hand, the canonical forms of $\Phi_{;\alpha\beta}$, (3.15a—b) can be represented in terms of the vectors of the null-leg as

$$\Phi_{;\alpha\beta} = \xi t_\alpha t_\beta - 2m({}_\alpha\overline{m}_\beta)\,\Theta - \sigma[m_\alpha m_\beta + \overline{m}_\alpha \overline{m}_\beta] =$$

$$= \xi t_\alpha t_\beta - m_\alpha(\Theta\overline{m}_\beta + \sigma m_\beta) - \overline{m}_\alpha(\Theta m_\beta + \sigma\overline{m}_\beta) \tag{4.3a}$$

$$\Phi_{;\alpha\beta} = -t_\alpha \eta g^{\hat{2}}_\beta + m_\alpha[i\eta t_\beta - \Theta g^{\hat{1}}_\beta] + C.C. \tag{4.3b}$$

Comparing it with (4.2) and applying the linear independence of t_α, m_α, and $\overline{m}_\alpha$, we conclude that in the first and the second case, respectively,

$$\xi_\alpha = \xi t_\alpha, \qquad \nu_\alpha = -(\Theta\overline{m}_\alpha + \sigma m_\alpha) \tag{4.4a}$$

and

$$\xi_\alpha = -\eta g^{\hat{2}}_\alpha, \qquad \nu_\alpha = Hg^{\hat{1}}_\alpha + i\eta t_\alpha. \tag{4.4b}$$

Thus, ξ_α, ν_α may be considered as known in both cases.

For our purposes we have to learns more about these vectors. We would like to know the properties of their derivatives; also, any information about η_α, μ_α will be of value.

From now on we restrict ourselves to the general case of $\Phi_{;\alpha\beta}$ with the canonical form (4.3a); all that follows, however, may be also repeated in the case of $\Phi_{;\alpha\beta}$ with the more pathological canonical form (4.3b).

The required information about η_α and μ_α can be obtained by studying the third derivatives of Φ.

Using the canonical form of $\Phi_{;\alpha\beta}$ as in (4.3a) and applying the definitions of $t_\alpha, \ldots, \overline{m}_\alpha$ and (4.1), one easily derives the following:

$$2s^\sigma \Phi_{;\sigma[\alpha\beta]} = t_{[\alpha\xi,\beta]} + m_{[\alpha}c_{\beta]} + \overline{m}_{[\alpha}\overline{c}_{\beta]} = t^\varrho s^\sigma R_{\varrho\sigma\alpha\beta}, \tag{4.5a}$$

$$2\overline{m}^\sigma \Phi_{;\sigma[\alpha\beta]} = -t_{[\alpha}c_{\beta]} + \overline{m}_{[\alpha}a_{\alpha]} + m_{[\alpha}b_{\alpha]} = t^\varrho \overline{m}^\sigma R_{\varrho\sigma\alpha\beta}, \tag{4.5b}$$

where we have introduced a_α, b_α, and c_α as abreviations for

$$a_\alpha = \Theta_{;\alpha} - \xi\Theta t_\alpha + [\Theta^2 + \sigma^2]\, s_\alpha,$$

$$b_\alpha = \sigma_{;\alpha} - \xi\sigma t_\alpha + 2\Theta\sigma s_\alpha - 2i\sigma\eta_\alpha,$$

$$c_\alpha = -(\Theta\overline{\mu}_\alpha + \sigma\mu_\alpha). \tag{4.6}$$

This enables to wr'te, as equivalent to $\Phi_{;\sigma\alpha\beta} - \Phi_{;\sigma\beta\alpha} = t_\varrho R_{\varrho\sigma\alpha\beta}$, the relation

$$t^\varrho R_{\varrho\sigma\alpha\beta} = 2t_\sigma\{t_{[\alpha}\xi_{;\beta]} + m_{[\alpha}c_{\beta]} + \overline{m}_{[\alpha}\overline{c}_{\beta]}\} + 2m_\sigma\{t_{[\alpha}c_{\beta]} - m_{[\alpha}b_{\beta]} - \overline{m}_{[\alpha}a_{\beta]}\} + C.C. \tag{4.7}$$

(Note that the right hand member, if skew-symmetrized over $\sigma\alpha\beta$, vanishes identically; this is consistent with $R_{\varrho[\sigma\alpha\beta]} = 0$.) These relations yield appreciable information about the unknown a_α, b_α, c_α and $\xi_{;\alpha}$.

Indeed, introduce the abreviations for the curvature quantities projected on the vectors (spinors), according to:

$$[uvRwu] = u^\alpha v^\beta R_{\alpha\beta\gamma\delta} w^\gamma z^\delta, \; (pqrs) = W_{ABCD} p^A q^B r^C s^D, \tag{4.8}$$
$$(\dot{p}\dot{q}rs) = U_{\dot{A}\dot{B}CD} p^{\dot{A}} q^{\dot{B}} r^C s^D.$$

Then, the following sequence of relations is easily deduced:

$$-\nabla_t \xi = 4[tsRts] = 8 \operatorname{Re}(ttss) - (\dot{t}\dot{s}ts) + \frac{R}{12}, \tag{4.9a}$$

$$(ta) = \nabla_t \Theta + \frac{1}{2}(\Theta^2+\sigma^2) = 4[t\bar{m}Rtm] = -\frac{1}{2}(\dot{t}\dot{t}tt), \tag{4.9b}$$

$$(tb) = (\nabla_t + \Theta - 2i(t\eta))\,\sigma = 4[t\bar{m}Rtm] = 4(tttt); \tag{4.9c}$$

then

$$(\bar{m}a)-(mb) = \nabla_{\bar{m}}\Theta - (\nabla_m - 2i(m\eta))\,\sigma = 4\,[t\bar{m}R\bar{m}m] = -4\,(ttts) - \frac{1}{2}(\dot{t}\dot{s}tt), \tag{4.10}$$

and

$$(tc) = 4\,[tsrt\bar{m}] = +\,4(ttts) - \frac{1}{2}(\dot{t}\dot{s}tt), \tag{4.11a}$$

$$(sc) = -\nabla_{\bar{m}} + 4\,[tsRs\bar{m}] = -\nabla_{\bar{m}} - 4(ssst)^* + \frac{1}{2}(\dot{s}\dot{s}st), \tag{4.11b}$$

$$(mc) = (sa) - 4\,[t\bar{m}Rsm] = \left(\nabla_s - \frac{1}{2}\,\xi\right)\Theta - 4\,[t\bar{m}Rsm] =$$
$$= \left(\nabla_s - \frac{1}{2}\,\xi\right)\Theta + 4\,(ttss) + \frac{R}{12}, \tag{4.11c}$$

$$(\bar{m}c) = (sb) - 4\,[t\bar{m}Rs\bar{m}] = \left(\nabla_s - \frac{1}{2}\,\xi - 2i(s\eta)\right)\sigma -$$
$$-4\,[t\bar{m}Rs\bar{m}] = \left(\nabla_s - \frac{1}{2}\,\xi - 2i(s\eta)\right)\sigma - \frac{1}{2}(\dot{s}\dot{s}tt). \tag{4.11d}$$

Of course, $\nabla_t = t^\alpha \nabla_\alpha$, $\nabla_s = s^\alpha \nabla_\alpha$, *etc.* These relations exhoust all information contained in (4.7), *i.e.*, they are equivalent to the "integrability conditions" $\Phi_{;\sigma\alpha\beta} - \Phi_{;\sigma\beta\alpha} = t^\varrho R_{\varrho\sigma\alpha\beta}$. Notice that the relation $(\bar{m}c)-(mc) = 4\,[tsR\bar{m}m]$ derivable directly from (4.5a) is not independent; it is a consequence of (4.11c) and $R_{\alpha[\beta\gamma\delta]} = 0$.

The first set of equations, (4.9), determines the derivatives of the optical scalars Θ, σ, and of the quasi-scalar ξ in the direction of the light rays. Of course, (4.9b) and (4.9c) imply the same as the (3.18) with $\varrho = 0$. But the imaginary part of (4.9c) tells us something more, namely that

$$\sigma \cdot (t\eta) = -2 \operatorname{Im}(\dot{t}\dot{t}tt) = -2 \operatorname{Im}[W_{ABCD} t^A t^B t^C t^D], \tag{4.12}$$

i.e., when $\sigma \neq 0$ it determines $(t\eta) = t^\alpha \eta_\alpha$ in terms of the conformal curvature.

The second set of equations, (4.10) and its complex conjugate contain some information about the "transverse" derivatives of Θ and σ as related to $(m\eta)$ and $(\overline{m}\eta)$. One can simply look on these relations as on equations which, when $\sigma\neq 0$, determine $(m\eta)$ and $(\overline{m}\eta)$ through the derivatives of Θ and σ and the curvature.

The relations (4.11) can be understood as equations determining entirely $c_\alpha = -\Theta\overline{\mu}_\alpha - \sigma\mu_\alpha$ in terms of the derivatives of the optical scalars and the curvature. Note, however, that according to (4.11d), the quantity $(\overline{m}c)$ depends on the still unknown $(s\eta)$. When c_α is known then, because of the standing assumption $\Theta^2-\sigma^2\neq 0$, the μ_α and $\overline{\mu}_\alpha$ are uniquely determined.

Now, a few words about the geometrical interpretation of the vectors $\omega_\alpha = 1/2\,(\xi_\alpha + i\eta_\alpha)$, ν_α and μ_α.

Consider the tetrad determined through the canonical form of $\Phi_{;\alpha\beta}$ at points x^α and $x^\alpha+\delta x^\alpha$ which are linked by an infinitesimal vector δx^α. The spinors which determine the tetrad $g^{\hat{\alpha}}_\alpha(x+\delta x)$ are, of course, $t_A(x)+t_{A,\,\alpha}(x)\,\delta x^\alpha$ and $s_A(x)+s_A(x)_{,\alpha}\,\delta x^\alpha$. Therefore, the spinors which determine the tetrad $[g^{\hat{\alpha}}_\alpha\,(x+\delta x)]_{||} = g'^{\hat{\alpha}}_\alpha$, *i.e.*, $g^{\hat{\alpha}}_\alpha\,(x+\delta x)$ parallely displaced from $x+\delta x$ to x along $-\delta x^\alpha$, clearly are

$$\left.\begin{aligned} t'_A &= (1+\delta\omega)t_A+\delta\nu s_A \\ s'_A &= \delta\mu t_A+(1-\delta\omega)s_A. \end{aligned}\right\}\text{, where } \delta\omega = \omega_\alpha\delta x^\alpha,\ \delta\nu = \nu_\alpha\delta x^\alpha,\ \delta\mu = \mu_\alpha\delta x^\alpha. \tag{4.13}$$

Now, (4.13) can be understood as an infinitesimal transformation of the elementary spinors t_A and s_A which on the level of the tetrad $g^{\hat{\alpha}}_\alpha$ induce an infinitesimal Lorentz transformation $g'^{\hat{\alpha}}_\alpha = g^{\hat{\alpha}}_\alpha + \delta L^{\hat{\alpha}}{}_{\hat{\beta}}\, g^{\hat{\beta}}_\alpha$.

In particular, taking $\delta x^\alpha = t^\alpha d\tau$ (a translation along the light ray) and the vectors ξ_α and ν_α in the form (4.4a) we have

$$\delta\xi = 0,\ \delta\nu = 0,\quad \delta\eta = (t\eta)d\tau,\ \delta\mu = (t\mu)d\tau \tag{4.14}$$

(of course, $\delta\omega = 1/2\,(\delta\xi+i\delta\eta)$; note that (4.14) is also valid when one applies (4.4b) instead of (4.4a)). Therefore, in this case (4.13) becomes

$$t'_A = \left(1+\frac{i}{2}\,(t\eta)\,d\tau\right)t_A \tag{4.15}$$

$$s'_A = \left(1-\frac{i}{2}\,(t\eta)d\tau\right)\,s_A+(t\mu)d\tau t_A.$$

The transformations generated by $(t\eta)$ and $(t\mu)$ may be understood as independent. It is obvious that the transformation induced by $(t\eta)$ corresponds to a *rotation* in $g^{\hat{1}}$, $g^{\hat{2}}$-plane through the angle

$$d\varphi = (t\eta)d\tau \rightarrow \frac{d\varphi}{d\tau} = (t\eta). \tag{4.16}$$

This rotation may also be comprehended as the rotation of the plane spanned on t^α and, say, $g^{\hat{1}}$ around t^α. This plane, as we shall see latter, can be in a sense identifyied with the plane of polarization, so that $d\varphi/d\tau$ describes the "rotation of the plane of polarization" per unit of the affine parameter. According to (4.12), when $\sigma\neq 0$, we have for this quantity the expression

$$\frac{d\varphi}{d\tau} = -\frac{2}{\sigma}\,\mathrm{Im}\,(W_{ABCD}t^A t^B t^C t^D), \tag{4.17}$$

so that when $\sigma\neq 0$ this rotation can occur only when induced by curvature. When $\sigma\neq 0$ and the space-time is flat $d\varphi/d\tau$ vanishes. It is of importance to strees the difference between $d\varphi/d\tau$ and the rotation ϱ in the case of a general congruency; ϱ corresponds to a rotation of the optical image as transferred by the beam of light rays. In the case of the potential congruency such a rotation is impossible. The $d\varphi/d\tau$ rotation, which may occur in the case of the potential congruency, consists in a rotation of the plane spanned by the space-like eigenvectors of the matrix $||\Phi^{;\alpha}{}_{;\beta}||$ when one moves along the light rays.

The transformation induced by $(t\mu)d\tau$ is less interesting; it of course, preserves the direction t_α but mixes $g^{\hat{1}}, g^{\hat{2}}$ with t_α : $g^{\hat{1}}_\alpha \to g^{\hat{1}}_\alpha + 2\,\mathrm{Re}\,(t\mu)d\tau t_\alpha$; $g^{\hat{2}}_\alpha \to g^{\hat{2}}_\alpha + 2\,\mathrm{Im}\,(t\mu)d\tau t_\alpha$. Hence it preserves the plane spanned on t_α and $g^{\hat{1}}_\alpha$, *i.e.*, anticipating the further developments, the "plane of polarization". It is rather a rotation inside of this plane. When $\sigma^2-\Theta^2\neq 0$ (our standing assumption) the relation $-\Theta\,(t\bar{\mu})-\sigma\,(t\mu) = 4(ttts)-\frac{1}{2}(\dot{t}\dot{s}tt)$ determines $(t\mu)$ as a linear expression in the curvature according to (4.11a). Hence, when $\Theta^2-\sigma^2\neq 0$ in flat space-time $(t\mu)$ necessarily vanishes.

Gathering this discussion we see that in flat space-time if $\Theta^2-\sigma^2\neq 0$ and $\sigma\neq 0$ the transformation (4.15) becames the identity transformation. This means that in the "flat" case and in presence of shear the tetrad to which the canonical form of $\Phi_{;\alpha\beta}$ is referred stays c o n s t a n t along each individual light ray.

Notice that along similar lines one can discuss (4.13) with δx^α differently specialized; it is natural to take δx^α along s_α, $g^{\hat{1}}_\alpha$, and $g^{\hat{2}}_\alpha$ and to see how different components of ω_α, ν_α, and μ_α influence the resulting transformation. That way one can give a full interpretation of all components of these vectors. We will not elaborate this discussion.

For the purpose of "vectorial" optics we will need some further differential identities. These can be obtained by the use of the Ricci formulae on the spinorial level:

$$\psi_{A;\,\alpha\beta} - \psi_{A;\,\alpha\beta} = \psi_B R^B_{A\alpha\beta}\,, \tag{4.18}$$

where the hybrid quantity $R^A_{B\alpha\beta}$ is related to the fundamental curvature spinors introduced in (3.19) by

$$R_{AB\alpha\beta} = \frac{1}{4}\,S_{\nu\mu AB}R^{\nu\mu}{}_{\alpha\beta}$$

$$= 2W_{ABCD}S_{\alpha\beta}{}^{CD} - \frac{1}{4}\,U_{\dot{C}\dot{D}AB}\,S_{\alpha\beta}{}^{\dot{C}\dot{D}} + \frac{R}{24}\,S_{\alpha\beta AB}\,. \tag{4.19}$$

This formula is valid under the assumption that the spinorial affine connections $\Gamma^A_{\alpha B}$ are traceles, *i.e.*, $\Gamma^A_{\alpha A} = 0$, or, physically, under the assumption that the electromagnetic potentials $A\mu$ do not participate in spinorial affine connections. (We assume the $A\mu's$ are not present in $\Gamma^A_{\alpha B}$ because the coupling of the studied fields described by equations (2.2) with the electromagnetic field seems to make very little sense.)

Specializing (4.18) for t_A and s_A and applying (4.1) one easily finds the "integrability conditions" of (4.1), *i.e.*, conditions that the right hand members of (4.1) from covariant derivatives, to be equivalent to

$$t^A t^B R_{AB\alpha\beta} = -2\, \nu_{[\alpha;\,\beta]} + 4\, \nu_{[\alpha}\omega_{\beta]} \tag{4.20a}$$

$$s^A s^B R_{AB\alpha\beta} = 2\, \mu_{[\alpha;\,\beta]} + 4\, \mu_{[\alpha}\omega_{\beta]} \tag{4.20b}$$

$$t^A s^B R_{AB\alpha\beta} = 2\, \omega_{[\alpha;\,\beta]} + 2\, \nu_{[\alpha}\mu_{\beta]}. \tag{4.20c}$$

Information contained in (4.20a) does not give anything new in comparison to what we already know from relations (4.9), (4.10), and (4.11). The equations (4.20b) contain new information about the derivatives of μ_α; on the level of the "vectorial optics" in higher orders it may be useful. The relations (4.20c) and their complex conjugates partly give the same implications as the consequences of $\Phi_{;\,\alpha\beta\gamma} - \Phi_{;\,\alpha\gamma\beta} = t^\varrho\, R_{\varrho\,\alpha\beta\gamma}$, but they also say something more.

Namely, contracting (4.20c) with $t^\alpha \overline{m}^\beta$, using (4.19) and (4.4a) for ξ_α and ν_α (also of course (4.1)) one obtains the identity

$$2i\nabla_{\overline{m}}(t\eta) - 2i\nabla_t(\overline{m}\eta) - i\Theta(m\eta) - \sigma(\overline{m}\eta) - 2\,(\overline{m}\eta)(t\eta) + 2i(t\eta)(\overline{\mu}t) - 2\sigma(\mu t) =$$

$$= 8W_{ABCD}t^A t^B t^C t^D \equiv 8\ (ttts); \tag{4.21}$$

this identity will be of importance in the study of the low orders of "optical development" on the level of "vectorial" optics.

V. The convenient Bais

After the discussion of "scalar" optics in two previous sections we would now like to prepare some tools which will enable us to conveniently approach the "vectorial" equations (2.4).

From the two spinors t_A and s_A normalized $t^A s_A = 1$ one can construct the quantities

$$\overset{l}{q}_{A_1\ldots A_{2s}} \underset{\mathrm{df}}{=} \exp\left[\frac{i\pi}{2}\,(2s-l)\right]\binom{2s}{l}^{1/2} s_{(A_1}\ldots s_{A_l}t_{A_{l+1}}\ldots t_{A_{2s})}, \tag{5.1a}$$

$$P_l^{A_1\ldots A_{2s}} \underset{\mathrm{df}}{=} \exp\left[\frac{i\pi}{2}\,(2s-l)\right]\binom{2s}{l}^{1/2} t^{(A_1}\ldots t^{A_l}s^{A_{l+1}}\ldots s^{A_{2s})}, \tag{5.1b}$$

where the index e runs through $0, 1, 2, \ldots\, 2s$ ($2s+1$ values). We will assume the summational convention with respect to this index.

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One easily checks that in consequence of these definitions

$$P_l^{A_1\ldots A_{2s}} \, q^l_{B_1\ldots B_{2s}} = \delta^{(A_1}{}_{(B_1} \ldots \delta^{A_{2s})}{}_{B_{2s})} \tag{5.2a}$$

$$P_k^{A_1\ldots A_{2s}} \, q^l_{A_1\ldots A_{2s}} = \delta^l_k \, ; \tag{5.2b}$$

$$q^{lA_1\ldots A_{2s}} = \exp\left[i\pi\,(s-l)\right] P_{2s-l}{}^{A_1\ldots A_{2s}}, \tag{5.3a}$$

$$p^{lA_1\ldots A_{2s}} = \exp\;\left[i\pi\,(s-l)\right] q^{2s-l}{}_{A_1\ldots A_{2s}}. \tag{5.3b}$$

It is clear that these quantities can be understood as the $2s+1$ orthonormal vectors related to the irreducible representation $D(s, o)$; a linear transformation between t_A and s_A (with the determinant equal one) induces the transformation of the q^l's according to $\|D(s, o)^k l\|$, the irreducible matrix corresponding to the spin s. Due to (5.3), the q^l's and P_l's may be considered as representing the same object.

One easily proves that

$$t^{A_1} q^l_{A_1\ldots A_{2s}} = \left(\frac{l}{2s}\right)^{1/2} q^{l-1}_{A_2\ldots A_{2s}}, \tag{5.4a}$$

$$s^{A_1} q^l_{A_1\ldots A_{2s}} = -i\left(\frac{2s-l}{2s}\right)^{1/2} q^l_{A_1\ldots A_{2s}}, \tag{5.4b}$$

which will be needed later.

If t_A and s_A are understood as fields $t_A(x)$ and $s_A(x)$ which have covariant derivatives as in (4.1), *i.e.*,

$$t_{A;\alpha} = \omega_\alpha t_A + \nu_\alpha s_A \tag{5.5}$$

$$s_{A;\alpha} = -\omega_\alpha s_A + \mu_\alpha t_A$$

then, applying the definitions (5.1) and (5.5), one finds without difficulty that

$$q^l_{A_1\ldots A_{2s}}{}_{;\alpha} = i\nu_\alpha \sqrt{(l+1)(2s-l)}\, q^{l+1}_{A_2\ldots A_{2s}} +$$

$$\omega_\alpha(2s-2l) q^l_{A_1\ldots A_{2s}} - i\mu_\alpha \sqrt{l(2s+1-l)}\, q^{l-1}_{A_1\ldots A_{2s}}. \tag{5.6}$$

All formulae of this section are valid for any arbitrary spinorial fields $t_A(x)$ and $s_A(x)$ normalized by $t^A s_A = 1$ which generate ω_α, ν_α, and μ_α according to (5.5).

It will be, however, instrumental in further developments to take t_A and s_A in (5.1) though (5.6) as spinors generating the tetrad to which the canonical form of $\Phi_{;\alpha\beta}$ is referred. Then, ω_α, ν_α, and μ_α became the quantities studied in the previous section with all the properties which we found.

VI. The scalar form of the eqations of "vectorial" optics

Now, we are sufficiently prepared to approach the main subject of this paper, *i.e.*, the study of the consequences of equations (2.4) of "vectorial" optics.

In these relations one can replace $\nabla^{\dot{A}B}\Phi$ by $t^{\dot{A}}t^B$. Moreover, using the directional derivatives

$$\nabla_t = t^\alpha \nabla_\alpha = \frac{1}{2}\, t_{\dot{A}}\, t_B \nabla^{\dot{A}B}, \;\ldots,\; \nabla_{\overline{m}} = \overline{m}^\alpha \nabla_\alpha = \frac{1}{2}\, s_{\dot{A}}\, t_B \nabla^{\dot{A}B}$$

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and the relation $\delta^A{}_B = t^A s_B - s^A t_B$ obvious from $t^A s_A = 1$, we have

$$\nabla^{\dot{A}B_1}\psi^{(n)}_{B_1\dots B_{2s}} = 2\,[t^{\dot{A}}(t^{B_1}\nabla_s - s^{B_1}\nabla_{\overline{m}}) + s^{\dot{A}}(s^{B_1}\nabla_t - t^{B_1}\nabla_m)]\,\psi^{(n)}_{B_1\dots B_{2s}}. \tag{6.1}$$

Using this in (2.4) and applying the linear independence of $t^{\dot{A}}$ and $s^{\dot{A}}$ we conclude that (2.4) splits into

$$t^{B_1}\psi^{(n+1)}_{B_1\dots B_{2s}} + 2\,(t^{B_1}\nabla_s - s^{B_1}\nabla_{\overline{m}})\,\psi^{(u)}_{B_1\dots B_{2s}} = 0, \tag{6.2a}$$

("pure" recurrence relations) and

$$(s^{B_1}\nabla_t - t^{B_1}\nabla_m)\,\psi^{(n)}_{B_1\dots B_{2s}} = 0, \tag{6.2b}$$

(integrability conditions for the recurrence relations).

In both these equations $n = -1, 0, 1, \dots$

Now, using (5.2a) we have

$$\psi^{(n)}{}_{A_1\dots A_{2s}} = \psi^{(n)}_l q^l{}_{A_1\dots A_{2s}}, \tag{6.3a}$$

where the "scalar amplitudes" (for the "parital waves" $\psi^{(n)}_{A_1\dots A_{2s}}$) $\psi^{(n)}_l$ are given by

$$\psi^{(n)}_l \overset{=}{\text{df}} P_l^{A_1\dots A_{2s}}\psi^{(n)}_{A_1\dots A_{2s}}. \tag{6.3b}$$

Now, considering t_A and s_A as determined with accuracy (3.7) and (3.9) by $t_\alpha = \Phi_{;\alpha} = \frac{1}{2}\, g_{\alpha\dot{A}B}t^{\dot{A}}t^B$ by the known (by assumption) eiconal function Φ we would like to derive a convenient set of equations for the "scalar amplitudes" $\psi^{(n)}$.

We will leave the detailed computations to the reader and give only the general outline of operations which have to be performed in order to obtain the desired equations.

After substituting (6.3a) into (6.2a—b) one obtains a sequence of relations where the differential operators act either upon $\psi^{(n)}_l$ or on q^l; in the last case we use (5.6) and express the directional derivatives of the q^l's through the q^l's and ν_α, ω_α, and μ_α projected on the direction of the differentiation. That way all terms become expressed through $q^l_{B_1\dots B_{2s}}$ with various l's. But, one still has to contract these quantities with t^{B_1} or s^{B_1}, as the structure of (6.2a—b) requires. These contractions may be executed with the help of (5.4a—b). Eventually, one ends up with some vanishing linear combinations of $q^l_{B_2\dots B_{2s}}$ (objects which correspond to spin $s - 1/2$ with $2s - 1$ indices). But these objects are linearly independent, therefore the scalar coefficients at these must separatly vanish. The most difficult part in these computations is the last part where one has to order the coefficients at $q^l_{B_2\dots B_{2s}}$ and to compute the numerical coefficients.

The final result is

$$\begin{aligned} \sqrt{(l+1)}\,\psi^{(n+1)}_{l+1} &= 2\,(\overline{m}\nu)\sqrt{l\,(2s+1-l)(2s-l)}\,\psi^{(n)}_{l-1} + \\ &-2i\sqrt{(2s-l)}\,[\nabla_{\overline{m}} + (\overline{m}\omega)(2s-2l) + (s\nu)(l+1)]\,\psi^{(n)}_l + \\ &-2\sqrt{(l+1)}\,[\nabla_s + (s\omega)(2s-2l-2) + (\overline{m}\mu)(2s-l)]\,\psi^{(n)}_{l+1} + \\ &+2\,(s\mu)\sqrt{(l+1)(l+2)(2s-1-l)}\,\psi^{(n)}_{l+2}; \end{aligned} \tag{6.4}$$

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these are the "pure" recurrence relations; remember $\psi_l^{(-1)} \equiv 0$. The numbers n and l run through: $n = 1, 0, 1...$; $l = 0, 1, ..., 2s-1$. Moreover,

$$\sqrt{2s-l}\,[\nabla_t+(t\omega)(2s-2l)+(l+1)(m\nu)]\,\psi_l^{(n)}+$$
$$-i\sqrt{l+1}\,[\nabla_m+(m\omega)(2s-2l-2)+(2s-l)(t\mu)]\,\psi_{l+1}^{(n)}+$$
$$-(m\mu)\sqrt{(l+1)(l+2)(2s-1-l)}\,\psi_{l+2}^{(n)} = 0; \tag{6.5}$$

these are the integrability conditions of the recurrency relations; here also $n = 0, 1, 2,...$ and $l = 0, 1, ..., 2s-1$.

In these formulae $(t\omega) = t^\alpha\omega_\alpha$, *etc.* The terms proportional to $(t\nu)$ are omitted because even in the case where t_A and s_A are only restricted by $t_\alpha = 1/2\, g_{\alpha\dot{A}B}\, t^{\dot{A}}t^B$ (the arbitrariness of transformations (3.7) and (3.9)) $(t\nu)$ must vanish.

These formulae become slinghtly simpler when one considers t_A and s_A to be fixed by the canonical form of $\Phi_{;\alpha\beta}$. According to (4.4a—b), *e.g.*, in the case of the canonical form (3.15a):$(s\nu) = 0$, Re $(t\omega) = 0$, *etc.*

VII. "Vectorial" optics in low orders in $1/ik$

The equations (6.4—5) in the general form look rather involved. In order to learn about their physical content we will study these relations for a few low values of n: $n = -1, 0, 1$. In doing so we will restrict ourselves entirely to $\Phi_{;\alpha\beta}$ of the canonical form (3.15a) and the spinors t_A, s_A related to it, so that all the machinery of Section IV will be applicable. (It presents no difficulty to repeat all that follows in the case of the more pathological form of $\Phi_{;\alpha\beta}$, (3.15b), but that will be left for the further).

With ω_α, ν_α, and μ_α related to t_A and s_A which generate the tetrad to which the canonical form (3.15a) is referred, relations (6.4) and (6.5) can be rewritten as

$$\sqrt{l+1}\,\psi_{l+1}^{(n+1)} = \sigma\sqrt{l(2s+1-l)\,(2s-l)}\,\psi_{l-1}^{(n)}+$$
$$-2i\sqrt{2s-l}\,[\nabla_{\overline{m}}+i(\overline{m}\eta)\,(s-l)]\,\psi_l^{(n)}-$$
$$-2\sqrt{l+1}\left[\nabla_s+\left(\frac{1}{2}\,\xi+i(s\eta)\right)(s-l-1)+(\overline{m}\mu)\,(2s-l)\right]\psi_{l+1}^{(n)}+$$
$$+2(s\mu)\sqrt{(l+1)(l+2)(2s-1-l)}\,\psi_{l+2}^{(n)}, \quad \begin{cases} n=-1, 0, 1, ... \\ l = \quad 0, 1, ..., 2s-1 \end{cases} \tag{7.1}$$

and

$$\sqrt{2s-l}\left[\nabla_t+i(t\eta)\,(s-l)+\frac{1}{2}\,(l+1)\Theta\right]\psi_l^{(n)}+$$
$$-i\sqrt{l+1}\,[\nabla_m+i(m\eta)\,(s-l-1)+(t\mu)\,(2s-l)]\,\psi_{l+1}^{(n)}-$$
$$-(m\mu)\sqrt{(l+1)(l+2)(2s-1-l)}\,\psi_{l+2}^{(n)} = 0 \tag{7.2}$$
$$\begin{cases} n = 0, 1, ... \\ l = 0, 1, ..., 2s-1. \end{cases}$$

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Now, in (7.1) set $n = -1$. Because $\psi_l^{(-1)} \equiv 0$; therefore,

$$\psi_l^{(0)} = 0, \qquad l = 1, 2, \dots 2s. \tag{7.3}$$

Hence, in the lowest order only $\psi_l^{(0)}$ is different from zero; it must be non-zero because otherwise due to (7.3) we would have $\psi^{(0)}_{A_1 \dots A_{2s}} = 0$ which cannot be.

But the scalar amplitudes of the lowest order must also obey the integrability conditions (7.2). Because of (7.3), Eq. (7.2) provides us with not-trivial condition when $n = 0$ only for $l = 0$. This condition is

$$\left[\nabla_t + is(t\eta) + \frac{1}{2}\,\Theta\right]\psi_0^{(0)} = 0, \tag{7.4}$$

i.e., it determines the variation of $\psi_0^{(0)}$ along light rays through the magnification Θ and $(t\eta)$, the quantities of the "scalar" optics.

Now, setting $n = 0$ in (7.1) and remembering that $\psi_{l+1}^{(0)} = 0$ we obtain

$$\begin{aligned}\sqrt{l+1}\,\psi_{l+1}^{(1)} &= \sigma\sqrt{l(2s+1-l)(2s-l)}\,\psi_{l-1}^{(0)} - \\ &-2i\sqrt{2s-l}\,[\nabla_{\bar{m}} + i(\bar{m}\eta)(s-l)]\,\psi_l^{(0)}.\end{aligned} \tag{7.5}$$

Substituting $l = 0, 1$ and $l \to l+2$ here, one obtains

$$\psi_1^{(1)} = -2i\sqrt{2s}\,[\nabla_{\bar{m}} + i(\bar{m}\eta)s]\,\psi_0^{(0)}, \tag{7.6a}$$

$$\psi_1^{(1)} = \sigma\sqrt{s(2s-1)}\,\psi_0^{(0)}, \tag{7.6b}$$

$$\psi_{l+2}^{(1)} = 0 \qquad \text{for } l \geqslant 1. \tag{7.6c}$$

Therefore, of all amplitudes $\psi_l^{(1)}$ of the first order only $\psi_0^{(1)}$, $\psi_1^{(1)}$, and $\psi_2^{(1)}$ may be different from zero; $\psi_2^{(1)}$ however vanishes whan 1) $s = 1/2$, which is only formal (in the case of neutrinos l runs only through $l = 0, 1$ anyway), and 2) when the shear σ tends to zero.

We still have to satisfy the integrability conditions of the first order. Set $n = 1$ in (7.2). As not-trivial conditions we get only those for $l = 0, 1, 2$:

$$\begin{aligned}&\sqrt{2s}\left[\nabla_t + i(t\eta)s + \frac{1}{2}\,\Theta\right]\psi_0^{(1)} - \\ &-i[\nabla_m + i(m\eta)(s-1) + 2s(t\mu)]\,\psi_1^{(1)} - (m\mu)\sqrt{2(2s-1)}\,\psi_2^{(1)} = 0\end{aligned} \tag{7.7a}$$

$$\sqrt{2s-1}\,[\nabla_t + i(t\eta)(s-1) + \Theta]\,\psi_1^{(1)} - i\sqrt{2}\,[\nabla_m + i\,(m\eta)(s-2) + (2s-1)(t\mu)]\,\psi_2^{(1)} = 0 \tag{7.7b}$$

$$\sqrt{2s-2}\left[\nabla_t + i(t\eta)\,(s-2) + \frac{3}{2}\,\Theta\right]\psi_2^{(1)} = 0. \tag{7.7c}$$

Expressing $\psi_1^{(1)}$ and $\psi_2^{(1)}$ through $\psi_0^{(0)}$ using (7.6), one obtains

$$\begin{aligned}\left[\nabla_t + i(t\eta)s + \frac{1}{2}\,\Theta\right]\psi_0^{(1)} &= \{2\{\nabla_m + i(m\eta)\,(s-1) + 2s(t\mu)]\,[\nabla_{\bar{m}} + i(\bar{m}\eta)s] + \\ &+ (2s-1)\,(m\mu)\sigma\}\,\psi_0^{(0)}\end{aligned} \tag{7.8}$$

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and

$$\sqrt{2s-1}\,\{2\,[V_t+i\,(t\eta)(s-1)+\Theta]\,[V_{\bar m}+i\,(\bar m\eta)\,s]+$$
$$+[V_m+i\,(m\eta)(s-2)+(2s-1)(t\mu)]\,\sigma\}\,\psi_0^{(0)} = 0 \tag{7.9a}$$

$$\sqrt{(2s-1)(s-1)}\,\left\{V_t+i(t\eta)\,(s-2)+\frac{3}{2}\,\Theta\right\}\sigma\psi_0^{(0)} = 0. \tag{7.9b}$$

The role of equation (7.8) is clear: it determines the variation of $\psi_0^{(1)}$ along "light rays" with transverse derivatives of $\psi_0^{(0)}$ as the sources of the equation.

Equations (7.9a—b) as they stand look rather strange; they seem to impose some further differential conditions on the amplitude $\psi_0^{(0)}$. But, this is not so. They, in fact, reduce to two simple algebraic conditions on the curvature related to $\Phi_{;\alpha\beta}$.

Indeed, examine first the simpler condition (7.9b). It may, of course, be rewritten as

$$\sqrt{(2s-1)\,(s-1)}\left\{\sigma\left(V_t+i(t\eta)\,(s-2)+\frac{3}{2}\,\Theta\right)+(V_t\sigma)\right\}\psi_0^{(0)} = 0. \tag{7.10}$$

But using here (7.4) we reduce it to:

$$\sqrt{(2s-1)(s-1)}\,\{(V_t\sigma)+\Theta\sigma-2i\,(t\eta)\,\sigma\}.\ \psi_0^{(0)} = 0. \tag{7.11}$$

Now, (4.9c) can be applied. We obtain the condition

$$\sqrt{(2s-1)(s\pm1)}\cdot W_{ABCD}t^At^Bt^Ct^D\cdot\psi_0^{(0)} = 0. \tag{7.12}$$

This algebraic condition has rather strong implications. It is automatically fulfilled for $s = 1/2$ and $s = 1$, *i.e.*, neutrinos and electrodynamincs. However for higher spins, because $\psi_0^{(0)}\neq 0$ by assumption, it leads to

$$s>1\rightarrow W_{ABCD}t^At^Bt^Ct^D = 0. \tag{7.13}$$

It may be fulfilled either when $W_{ABCD} = 0$ (the conformally flat V_4) or when $W_{ABCD}\neq 0$ with t_A proportional to one of the 4 (in general [1—1—1—1] case) Penrose spinors. In other words, the geodesic null vector $t_\alpha = \Phi_{;\alpha}$ must be proportional to one of the Debever's vectors [16], [17], [9]. If this is so, the corresponding Debever's vector has to be a geodesic itself, which imposes rather severe restrictions on the curvature of space-time. Note that all of the trouble with the condition discussed becomes immaterial in the case of a shear free congruency, as one can see directly from (7.9b).

Notice that condition (7.9b) equivalent to (7.13) appears to possess some relationship with the Sachs-Goldberg theorem [18], [19].

In an algebraically special V_4 with a shear-free congruency the condition (7.9b) is trivally fulfilled.

Take now the condition (7.9b). In order to reduce it to an algebraic condition observe first that by using (4.1) one can easily prove that

$$V_tV_{\bar m}-V_{\bar m}V_t = (t\bar\mu)V_t+\frac{1}{2}\,(2i(t\eta)-\Theta)V_{\bar m}-\frac{1}{2}\,\sigma V_m. \tag{7.14}$$

Using this in (7.9a) and aliminating afterwards the derivatives $\nabla_t \psi_0^{(0)}$ with the help of (7.4) one after ordering obtains

$$\sqrt{2s-1}\,\{\langle (tc)-[\nabla_{\bar{m}}\Theta-\nabla_m\sigma+2i(m\eta)\,\sigma]\rangle + s\langle -2i\nabla_{\bar{m}}(t\eta)+2i\nabla_t(\bar{m}\eta)-2i(t\bar{\mu})(t\eta)+$$
$$+2(t\eta)\,(\bar{m}\eta)+i\Theta(\bar{m}\eta)+i\sigma(m\eta)+2(t\mu)\,\sigma\rangle\}.\,\psi_0^{(0)} = 0. \tag{7.15}$$

All derivatives acting on $\psi_0^{(0)}$ have been cancelled, so that this condition is purely algebraic. The c_α entering in the form (tc) in this formula is the same as defined that by (4.6). Now, in the first line we may use the identities (4.10) and (4.11a) while all the terms proportional to the spin s can be replaced by $-8s\,(ttts)$ according to (4.21). In effect, (7.15) is equivalent to condition (7.9a) and reduces to the following simple condition on the conformal curvature:

$$\sqrt{2s-1}\;(s-1)\;W_{ABCD}t^At^Bt^Cs^C\cdot\psi_0^{(0)} = 0. \tag{7.16}$$

Again, in the case of neutrinos and electromagnetic waves ($s = 1/2,\ 1$) this condition is automatically fulfilled. Because $\psi_0^{(0)}\neq 0$ by assumption, for higher spins we have

$$s<1\rightarrow W_{ABCD}t^At^Bt^Cs^C = 0. \tag{7.17}$$

The condition (7.17) must be valid simultaneuly with (7.13) for any $s>1$. Therefore it does not say anything essentially new in comparison with (7.13). If the V_4 is conformally flat or it t^A is proportional to one of Penrose's spinors, both conditions are fulfilled. (There is however the possibility of satisfying (7.17) when s_A is proportional to one of Penrose's spinors while t_A is not; this possibility has to be rejected because by accepting it we would violate (7.13) which must hold simultaneusly with (7.17).

VIII. Plane waves, planes of polarization

The aim of this section is to recall some fundamental facts about plane waves, as expressed in the "spinorial language". Moreover, we will introduce here the notion of planes of polarization as related with plane waves of arbitrary spins.

This will enable us to develop conveniently in the next section the physical interpretation of the results of "vectorial optics".

For simplicity, we restrict ourselves in this section to special relativity ($V_4\rightarrow S_4$). Later on, however, we will return to V_4, transfering to V_4 the physical interpretation clearly established in S_4.

The wave equations (2.2) considered as equations in carlesian coordinates in S_4 are simply

$$\partial^{AB_1}\psi_{B_1\ldots B_{2s}} = 0. \tag{8.1}$$

Now, to every spinor t_A and the number $\varepsilon = \pm 1$, one can attribute a solution to (8.1)

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of the type of a plane circularlypolarized wave:

$$\psi_{A_1 \ldots A_{2s}}(x^\alpha, t^B, k, \varepsilon) = a_s(t^B, k, \varepsilon)\, t_{A_1} \ldots t_{A_{2s}} \exp[ik\Phi] \tag{8.2a}$$

$$\Phi = \varepsilon t_\alpha x^\alpha; \; t_\alpha = \frac{1}{2} g_{\alpha \dot{A} B} t^{\dot{A}} t^B \qquad t_0 > 0, \; t^\alpha t_\alpha = 0. \tag{8.2b}$$

In this formula $a_s(t^B, k, \varepsilon)$ with the same dimensions as $\psi_{A_1 \ldots A_{2s}}$ is some complex scalar amplitude; the spinor t_A is dimensionless, so it is the null vector of the propagation, t_α. The phase Φ has the dimension of length. The number k of dimension (length)$^{-1}$ secures the dimensionless character of $k\Phi$. The frequency of the temporal oscillations is clearly $\omega = \varepsilon c k t_0$, so that $\varepsilon = \pm 1$ determines the sign of the frequency.

One easily proves that (8.2a) represents the general form of a solution of (8.1) which depends on coordinates only through the factor $\exp(ik\Phi)$, Φ being linerar in the coordinates. We used the expression "circularly polarized wave" anticipating the interpretation which will be given later.

By superimposing waves with positive and negative frequencies but the same k and t_A one obtains the waves in the more general "states of polarization":

$$\psi_{A_1 \ldots A_{2s}}(x^\alpha, t^B, k) \underset{\mathrm{df}}{=} Z_s(x^\alpha, t^B, k)\, t_{A_1} \ldots t_{A_{2s}}, \tag{8.3a}$$

where

$$Z_s(x^\alpha, t^B, k) = a_s(t^B, k, 1) e^{ikt_\alpha x^\alpha} + a_s(t^B, k, -1) e^{-ikt_\alpha x^\alpha}. \tag{8.3b}$$

For half-integer values of s the waves (8.2) with $\varepsilon = 1$, correspond to positive energy density (particles) while those with $\varepsilon = -1$ to negative energy density (antiparticles). Hence, for the half integral spin the waves (8.3) represent some mixtures of "particles" with "anti-particles" which have the common propagation vector $k_\alpha \underset{\mathrm{df}}{=} k\, t_\alpha$.

The wave (8.3a) can be represented as:

$$\psi_{A_1 \ldots A_{2s}}(x^\alpha, t^B, k) = \zeta_{(r)A_1} \ldots \zeta_{(r)A_{2s}} \tag{8.4a}$$

where, if

$$Z_s = R_s e^{iA_s}, \; R_s > 0 \tag{8.4b}$$

then,

$$\zeta(r)A \underset{\mathrm{df}}{=} [R_s]^{1/2s} t_A \exp\left[\frac{i}{2s}(A_s + 2\pi r)\right]; \; r = 0, 1, \ldots, 2s-1. \tag{8.4c}$$

The factor $\exp(i\pi r/s)$ $(r = 0, 1, \ldots, 2s-1)$ describes all the arbitrariness in the definition of ζ_A through the plane wave $\psi_{A_1 \ldots A_{2s}}(x^\alpha, t^B, k)$. We have $2s$ of the possible "branches" at our disposal.

The spinor $\zeta_{(r)A}$ obeys the neutrino equation $\partial^{\dot{A}B} \zeta_{(r)B} = 0$. The fact that can construct plane waves with higher spins from the plane waves of neutrinos forms the starting point of the de Broglie's theory of fussion [20], [21].

We will apply this fact to a much simpler purpose. Exploiting it, we will be able to attribute to the general waves (8.3a) some well-defined "planes of polarization".

Indeed, from the well-defined spinor $\zeta_{(r)A}$ one can construct higher irreducible objects. The simplest is the $D(1/2,\ 1/2)$ object

$$\zeta_{(r)\alpha} \underset{\mathrm{df}}{=} \frac{1}{2}\, g_{\alpha\dot{A}B}\zeta_{(r)}{}^{\dot{A}}\zeta_{(r)}{}^{B} = |Z_s|^{1/s}t_\alpha, \tag{8.5}$$

where the right hand side is the same for all branches. Much more interesting is the $D(0,\ 1)$ object $\zeta_{(r)A}\ \zeta_{(r)B}$ which has a self-dual bi-vector as its tensorial image. From the last, one can construct a real bi-vector (a null object) and its dual:

$$f_{(r)\alpha\beta} \underset{\mathrm{df}}{=} S_{\alpha\beta}{}^{AB}\,\zeta_{(r)A}\,\zeta_{(r)B} + \mathrm{C.C} \tag{8.6a}$$

$$\check{f}_{(r)\alpha\beta} \underset{\mathrm{df}}{=} S_{\alpha\beta}{}^{AB}\zeta_{(r)A}\zeta_{(r)B} - \mathrm{C.C.} \tag{8.6b}$$

The null bi-vector (8.6a) for $s = 1$ coincides with the null electromagnetic field (a plane wave) which is the tensorial image of $\psi_{A_1A_s}(x^\alpha,\ t^B,\ k)$. Both branches $r = 0{,}1$ generate the same object, *i.e.*, $f_{(0)\alpha\beta} = f_{(1)\alpha\beta}$ for $s = 1$.

But, the object (8.6a) is also well-defined for any value of the spin s with an accuracy up to the choice of the branch number r. Therefore, one can formally proceed with the interpretation of $f_{(r)\alpha\beta}$ as it were the electromagnetic field tensor for any value of s. This way the notions, familiar in the electromagnetic case, of circular, linear and eliptic polarization can be extended in a natural fashion for all values of s.

Further on, we will refer to $f_{(r)\alpha\beta}$ as to quasielectromagnetic field when $s \neq 1$. For $s = 1$ it becames a genuine electromagnetic field tensor.

According to (8.4c) the definitions (8.6a—b) may be rewritten as

$$f_{(r)\alpha\beta} = [R_s]^{1/s}e^{i\Omega_{s,r}}\, S_{\alpha\beta}{}^{AB}t_At_B + \mathrm{C.C} \tag{8.7a}$$

$$\check{f}_{(r)\alpha\beta} = \text{as above } - \mathrm{C.C,} \tag{8.7b}$$

where

$$\Omega_{s,r} \underset{\mathrm{df}}{=} (\Lambda_s + 2\pi r)/s. \tag{8.7c}$$

Now, take a time-like vector $g_\alpha^{\hat{0}}$ pointing into the future and normalized to unity: $g^{\nu\mu}g_\nu^{\hat{0}}g_\mu^{\hat{0}} = 1$. Using it, one can define quasi-"electric" and quasi-"magnetic" 4-vectors by

$$E_{(r)\alpha} \underset{\mathrm{df}}{=} -f_{(r)\alpha\beta}g^{\hat{0}\beta}, \qquad \check{H}_{(r)\alpha} \underset{\mathrm{df}}{=} -i\check{f}_{(r)\alpha\beta}g^{\hat{0}\beta}. \tag{8.8}$$

These vectors may be represented as $g_\alpha^{\hat{0}} = t_\alpha + s_\alpha$ where $t^\alpha s_\alpha = \frac{1}{2}$, $s^\alpha s_\alpha = 0$ implies $s_\alpha = \frac{1}{2}\, g_{\alpha\dot{A}B}s^{\dot{A}}s^B$, $t^As_B = 1$. In other words, with fixed spinor t_A the spinor s_A is determined by the given choice of $g_\alpha^{\hat{0}}$. On the spinors t_A and s_A we span the null-leg with the related tetrad — the formalism of (3.6)—(3.13). The arbitrariness in the choice of $g_\alpha^{\hat{0}}$ corresponds to the transformations of elementary spinors given by

$$t_A \to t_A' = t_A; \quad s_A \to s_A' = s_A + \beta t_A. \tag{8.9}$$

One easily checks that $s_{\alpha\beta}{}^{AB}t_A t_B = 8t_{[\alpha}\overline{m}_{\beta]}$. Considering $g_\alpha^{\hat{0}}$ as given, *i.e.*, t_A and s_A fixed, one easily finds that

$$f_{(r)\alpha\beta} = 4t_{[\alpha}E_{(r)\beta]}, \quad \check{f}_{(r)\alpha\beta} = -4it_{[\alpha}\check{H}_{(r)\beta]}, \tag{8.10}$$

with

$$E_{(r)\alpha} = 2(R_s)^{1/s}\{\cos\Omega_{s,r}g_\alpha^{\hat{1}} + \sin\Omega_{s,r}g_\alpha^{\hat{2}}\} \tag{8.11a}$$

$$H_{(r)\alpha} = 2(R_s)^{1/s}\{-\sin\Omega_{s,r}g_\alpha^{\hat{1}} + \cos\Omega_{s,r}g_\alpha^{\hat{2}}\}. \tag{8.11b}$$

Now, the objects $f_{(r)\alpha\beta}$ and $\check{f}_{(r)\alpha\beta}$ determine the quasi-"electric" and quasi-"magnetic" plane of polarization.

These planes are defined as follows (with the understanding that $f_{(r)\alpha\beta}$ and $\check{f}_{(r)\alpha\beta}$ are taken at a fixed event $\overset{\circ}{x}{}^\alpha$):

$$\Pi_{(r)E} : f_{(r)}{}^{[\alpha\beta}x^{\gamma]} = 0, \tag{8.12a}$$

$$\Pi_{(r)H} : \check{f}_{(r)}{}^{[\alpha\beta}x^{\gamma]} = 0,$$

which is equivalent to

$$\Pi_{(r)E} : \check{f}_{\alpha\beta}x^\beta = 0, \quad \Pi_{(r)H} : f_{\alpha\beta}x^\beta = 0. \tag{8.12b}$$

With $f_{(r)\alpha\beta}$ and $\check{f}_{(r)\alpha\beta}$ taken in the form of (8.10), one can represent these planes parametrically as

$$\Pi_{(r)E}: x^\alpha = \mu t^\alpha + \nu E_{(r)}{}^\alpha, \tag{8.12c}$$

$$\Pi_{(r)H}: x^\alpha = \mu t^\alpha + \nu \check{H}_{(r)}{}^\alpha,$$

where μ and ν are real parameters.

As it is clear from (8.12a—b) these planes are entirely determined by the original object $\zeta_{(r)A}\,\zeta_{(r)B}$ and do not depend on the choice of $g_\alpha^{\hat{0}}$ which was instrumental in the definitions of $E_{(r)\alpha}$ and $H_{(r)\alpha}$. One can easily see that a new choice of $g_\alpha^{\hat{0}}$ which corresponds to the transformation (8.9) changes $E_{(r)\alpha}$ and $\check{H}_{(r)\alpha}$ by only some additive terms proportional to t_α, so that (8.10) remains true with so re-defined $E_{(r)\alpha}$ and $\check{H}_{(r)\alpha}$. It is also clear that such re-definitions preserve the parametric representation of our planes (8.12c). The vectors $E_{(r)\alpha}$ and $\check{H}_{(r)\alpha}$ are orthogonal. Therefore, the planes $\Pi_{(r)E}$, $\Pi_{(r)H}$ are orthogonal. All these planes contain, of course, the propagation vector t_α.

The question arises concerning what happens with the planes when one moves from the point $\overset{0}{x}{}^\alpha$ to which they are attached to the neighboring point $\overset{0}{x}{}^\alpha + dx^\alpha$.

Of course, the propagation vector t_α stays unchanged. But as is clear from (8.11a—b) and (8.12c), the planes attached at $\overset{0}{x}{}^\alpha + dx^\alpha$ can be understood as the same planes as were attached to $\overset{0}{x}{}^\alpha$ after the rotation around t_α through the infinitesimal angle

$$dx^\alpha\partial_\alpha\Omega_{s,r} = \frac{1}{s}\,dx^\alpha\partial_\alpha A_s \underset{\mathrm{df}}{=} d\Omega_s; \tag{8.13}$$

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this angle happens to be the same for all branches. With the help of (8.4b), (8.3b) this quantity can be computed, and in obvious abbreviated notation it is

$$d\Omega_s = \frac{k}{s}\, t_\alpha dx^\alpha \cdot \frac{|a_s(1)|^2 - |a_s(-1)|^2}{|Z_s|^2}. \tag{8.14}$$

This formula can be rewritten in a more plausible form. Indeed, let

$$a_s(1) = pe^{-isu+iv}, \quad a_s(-1) = qe^{-isu-iv}, \tag{8.15a}$$

where $p \underset{\text{df}}{=} |a_s(1)|$, $q \underset{\text{df}}{=} |a_s(-1)|$. Moreover, let:

$$\lambda \underset{\text{df}}{=} kt_\alpha x^\alpha + v, \qquad \epsilon \underset{\text{df}}{=} \frac{2pq}{p^2+q^2}, \qquad \varepsilon \underset{\text{df}}{=} \text{sign}\,(p-q). \tag{8.15b}$$

In principle, all quantities so defined should have subscripts s; we omit them for simplicity.

In terms of these convenient quantities one can rewrite (8.14) as

$$s\Omega_s = \frac{\varepsilon k}{s} \cdot t_\alpha dx^\alpha \cdot \frac{1-\epsilon^2}{1+\epsilon \cos 2\lambda}. \tag{8.16}$$

This formula forms a convenient starting point for the physical interpretation of our waves and their related planes of polarization.

Notice first that when one moves along the "light ray" *i.e.*, $dx^\alpha = t^\alpha d\tau$, due to $t^\alpha t_\alpha = 0$, $d\Omega_s$ stays equal to zero. Therefore, along light rays (4 dimensionally) our planes stay constant. The same holds when one moves along any direction orthogonal to the null propagation vector, *i.e.*, when $t_\alpha dx^\alpha = 0$.

The factor $t_\alpha dx^\alpha$ may be different from zero when dx^α contains a component along s^α, in particular when one moves along the temporal direction $g^{\hat{0}}_\alpha$. With $dx^\alpha = g^{\hat{0}\alpha}\, d\tau$, Eq. (8.16) reduces to:

$$\frac{d\Omega_s}{d\tau} = \frac{\varepsilon k}{2s} \cdot \frac{\sqrt{1-\epsilon^2}}{1+\epsilon \cos 2\lambda}. \tag{8.17}$$

This gives rise to a few distinguished situations. First when either $a_s(1)$ or $a_s(-1)$ vanishes so that we are dealing with a wave with the definite sign of frequency then $\epsilon = 0$, and Eq. (8.17) reduces to

$$\frac{d\Omega_s}{d\tau} = \frac{\varepsilon k}{2s}. \tag{8.18}$$

Therefore, ours planes of polarization rotate with the constant "angular velocity" given by (8.18). The ε introduced by (8.15b) coincides here with the ε determining the sign of the frequency. For this reason we used the same symbol for both quantities.

The waves with property (8.18) will be called "circularly polarized waves".

Secondly, when $\epsilon = 1$, which is equivalent to $|a_s(1)| = |a_s(-1)|$, we have

$$\frac{d\Omega_s}{d\tau} = 0, \tag{8.19}$$

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so that our planes of polarization stay constant even when one moves in the temporal direction.

The waves with this property will be called "linearly polarized waves".

Finally, when $0<\epsilon<1$ the "nagular velocity" $\frac{d\Omega_s}{d\tau}$ reaches the minimal and maximal values when

$$\lambda = \pi \cdot n \to \frac{d\Omega_s}{d\tau} = \frac{\varepsilon k}{2s}\sqrt{\frac{1-\epsilon}{1+\epsilon}}, \quad \lambda = \left(n+\frac{1}{2}\right)\pi \to \frac{d\Omega_s}{d\tau} = \frac{\varepsilon k}{2s}\cdot\sqrt{\frac{1+\epsilon}{1-\epsilon}}. \tag{8.20}$$

The waves where the "angular velocity" $\frac{d\Omega_s}{d\tau}$ passes through maxima and minima (with increasing time in $\lambda = kt_\alpha x^\alpha + v$) will be called elliptically polarized. The discussion above was concerned only with the properties of the planes $\Pi_{(r)E}$, $\Pi_{(r)H}$ determined entirely by $\zeta_{(r)A}\zeta_{(r)B}$. However, when the temporal direction $g^{\hat{0}}_{\alpha}$ is given an the vectors $E_{(r)\alpha}$ and $\check{H}_{(r)\alpha}$ are well-defined, one can give an even more direct interpretation of the "states of polarization" defined on the level of the "planes of polarization".

Indeed, consider the projections of $E_{(r)\alpha}$ on $g^{\hat{1}}_{\alpha}$ and $g^{\hat{2}}_{\alpha}$, which according to (8.11a) are

$$x'_{(r)} = 2(R_s)^{1/s}\cos\Omega_{s,r}, \quad y'_{(r)} = 2(R_s)^{1/s}\sin\Omega_{s,r}. \tag{8.21}$$

Now, using the notation introduced by (8.15a) and (8.15b), one esaily finds that if $e^{iu}[x'_{(r)}+iy'_{(r)}] = x_{(r)}+iy_{(r)}$; then,

$$z_{(r)} \underset{\text{df}}{=} x_{(r)}+iy_{(r)} = \varrho e^{i\varphi_r} \tag{8.22a}$$

$$\varrho = 2[(p+q)^2\cos^2\lambda+(p-q)^2\sin^2\lambda]^{1/2s}, \tag{8.22b}$$

$$\varphi_r = \left[\operatorname{arctg}\left(\varepsilon\sqrt{\frac{1-\epsilon}{1+\epsilon}}\operatorname{tg}\lambda\right)+2\pi r\right]/s$$

This can be understood as the parametric representation of the curve $z_{(r)}(\lambda)$ plotted in the complex plane by the vector $E_{(r)\alpha}$ when x° changes and x^a stay constant. The curve corresponding to $\check{H}_{(r)\alpha}$ has the same representation but with φ_r shifted through $\pi/2$.

Notice that by eliminating λ from (8.22b) one derives the same relation for all branches of $E_{(r)\alpha}$:

$$\left(\frac{1}{2}\varrho\right)^{2s} = \frac{(p^2+q^2)(1-\epsilon^2)}{1-\epsilon\cos 2s\varphi_r}. \tag{8.23}$$

We would like to examine the shape of these "quasi-electric" and "quasi-magnetic" curves.

First, take $s = 1/2$. Here we have only one branch, $r = 0$. As one can sese from (8.23) the quasi-electric curve is an ellipse with the focus at the origin of coordinates. Fig. 1 xeplains the situation. The values $\varepsilon\lambda$ which correspond to the maximal and the minimal length of

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E are as indicated in the picture. Of course, ϱ max $= 2(p+q)^2$ and ϱ min $= 2(p-q)^2$. When either p or q vanishes both "magnetic" and "electric" ellipses tend to a circle (circular polarization). When $p = q$, the case of linear polarization, these curves degenerate into two lines (see Fig. 1a).

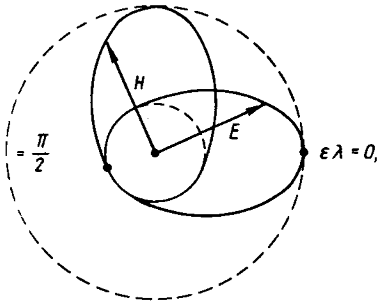

Fig. 1

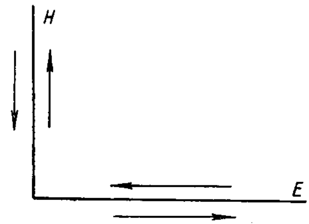

Fig. 1a

Now, consider the genuine electromagnetic case, $s = 1$. Here (8.22) can be seen to reduce to

$$z_{(r)} = 2(p+q)\cos\lambda + i2(p-q)\sin\lambda. \tag{8.24}$$

Hence, we also have an ellipse, but this time with the center (not the focus as for $s = 1/2$) at the origin of coordinates. Of course, both branches $r = 0, 1$ coincide here: $E_{(\sigma)\alpha} = E_{(1)\alpha}$. The situation is illustrated by Fig. 2.

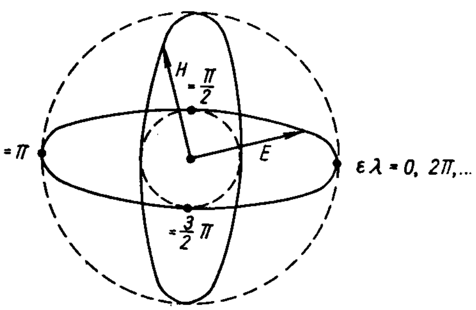

Fig. 2

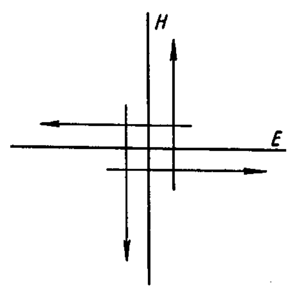

Fig. 2a

The maximal and minimal values of E and H are, of course, $2(p+q)$ and $2|p-q|$. E takes these values for the values of $\varepsilon\lambda$ given on the picture.

When either p or q vanishes we have circular polarization: both ellipses became one curve, a circle. When $p = q$ we have linear polarization (see Fig. 2a).

In the case of $s = 3/2$, E has three branches corresponding to $r = 0, 1, 2$. The vectors $E_{(r)\alpha}$ circulate around the common curve (see Fig. 3). The vectors E_0, E_1, and E_2 are indicated for the "initial" situation $\lambda = 0$. The values of $\varepsilon\lambda$ around the circle correspond to the "history" of the fundamental branch $E_{(0)\alpha}$ when it rotates around the curve. One

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obtains the "quasi-magnetic" curve for the three branches $H_{(r)\alpha}$ from Fig. 3 by rotating it through $\pi/2$. The radii of the external and the internal circles are obviously $2(p+q)^{2/3}$ and $2|p-q|^{2/3}$. When either p or q vanishes the "magnetic" and "electric" curves became one, a circle (circular polarization).

But, when $p = q$ our curve degenerates according to the scheme of Fig. 3a. The arrows describe the history of the fundamental branch $E_{(0)\alpha}$. E_0 first decreases along $\varphi = 0$ to

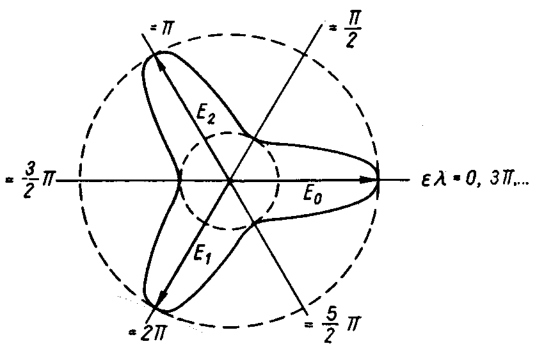

Fig. 3

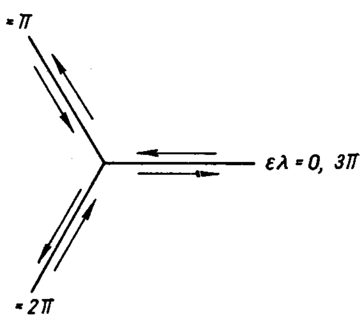

Fig. 3a

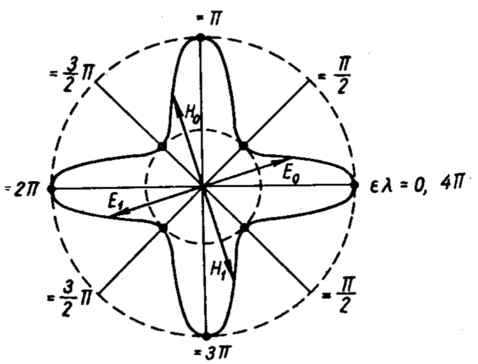

Fig. 4

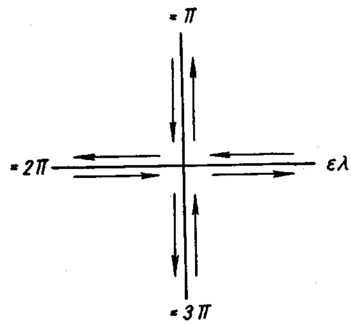

Fig. 4a

the value zero, then climbs along $\varphi = 2\pi/3$ to its maximal value, returns along this line to the origin, *etc.* It is clear however that the three constant planes $\Pi_{(r)E}$ are well-defined at all times. In that sense, we have linear polarization.

The last case which deserves detailed discussion is the case of $s = 2$.

As is clear from (8.22b), the branches E_0 and E_2 coincide as do E_1 and E_3; in reality E_0 and E_1 "anti-coincide": $E_{(0)\alpha} = -E_{(1)\alpha}$. The same is true with respect to $H_{(r)\alpha}$. Now, in all of the previous cases the "electric" and "magnetic" curves were different except for the degenerate case of circular polarization. For spin $s = 2$, due to exceptionally high symmetry, these curves a l w a y s coincide. The picture generalizing the previous ones is Fig. 4. The radii of the external and internal circles are respectively $2|p+q|^{1/2}$ and $2|p-q|^{1/2}$. The values of $\varepsilon\lambda$ corresponding to the maximas and minimas are indicated for the fundamental branch $E_{(0)\alpha}$. When either p or q vanishes we have circular polarization and the curve becames a circle. When $p = q$ the curve from Fig. 4 degenerates into that from Fig. 4a. The arrows illustrate the history of the fundamental branch $E_{(0)\alpha}$.

In a way, Fig. 4a, is similar to Fig. 2a.

It is clear from this discussion that for $s = 2$ we have only two different planes of polarization, similar to the case $s = 1$. Independent of the "history" of $E_{(0)\alpha}$, when $p = q$ these planes stay constant; in this sense we have here linear polarizarion.

Now, for arbitrary values of s the structure of the corresponding "electric" and "magnetic" curves is similar to that for $s = 3/2, 2$. It is clear from (8.22a—b) that when $\varepsilon\lambda = 0$

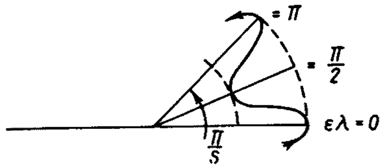

Fig. 5

the point $z_{(0)}$ lies on the x axis at the maximal possible distance from the origin of coordinates, *i.e.*, at ϱ max $= 2(p+q)^{1/s}$. When $\varepsilon\lambda$ increases to $\pi/2$, ϱ becames ϱ min $= 2|p-q|^{1/s}$. During this transition φ increases from 0 to $\pi/2s$. Then, when $\varepsilon\lambda$ increases to π one returns to ϱ max with $\varphi = \pi/s$. The story then repeats itself (see Fig. 5). The mechanism for both types of degeneration is obvious. When either p or q vanishes the curve becomes a circle; $p = q$ it degenerates into $2s$ radii which divide the circle into $2s$ symmetric segments.

IX. Physical interpretation

The aim of this Section is to examine the physical interpretation of results derived on the level of the "vectorial optics".

According to Section VII the beginning of the "optical series" has the form

$$\psi_{A_1 \ldots A_{2s}} = \left\{\psi_0^{(0)} q^0_{A_1 \ldots A_{2s}} + O\left(\frac{1}{ik}\right)\right\} \exp\{ik\Phi\} = \left\{e^{i\pi s}\psi_0^{(0)} t_{A_1} \ldots t_{A_{2s}} + O\left(\frac{1}{ik}\right)\right\} \exp\{ik\Phi\}. \tag{9.1}$$

The eiconal function Φ can be assumed to be a known fixed solution of $\Phi^{;\alpha}\,\Phi_{;\alpha} = 0$. Then t_A and s_A can be understood as fixed by the canonical form of $\Phi_{;\alpha\beta}$. As far as $\psi_0^{(0)}$ is concerned we are in possession of the information that

$$\left[\nabla_t + is(t\eta) + \frac{1}{2}\,\Theta\right] \psi_0^{(0)} = 0. \tag{9.2}$$

Comparing (9.1) and (8.2), we conclude that the beginning of the "optical series" has the algebraic structure of a plane, circularly polarized wave. In fact, it can be understood as such a wave locally. In the neighborhood of event x^α at $x^\alpha + \delta x^\alpha$ we have $\Phi(x + \delta x^\alpha) \approx \Phi(x) + t_\alpha \delta x^\alpha$; moreover, because t_α is geodesic it is covariantly constant along light rays. The "local" frequency of such a wave is clearly kct_0; because $t_0>0$, the sign of the frequency is determined by the sign of $k = \varepsilon|k|$ where $\varepsilon = \pm 1$.

By superimposing waves with positive and negative frequencies (both scalar amplitudes $\psi_0^{(0)}(+)$, $\psi_0^{(0)}(-)$ must obey (9.2)) one can obtain waves in more general states of polarization. All of the discussion in the previous section applyies to such waves, but only on the local scale.

Equation (9.2) integrated along the "light ray" yields

$$\psi_0^{(0)} = a\cdot\exp\left[-\int^{\tau} d\tau\left\{is(t\eta)+\frac{1}{2}\,\Theta\right\}\right], \tag{9.3}$$

where a is some function constant along the light ray, $\nabla_t a = 0$.

Therefore, if we restrict ourselves to the circularly polarized waves the quantity Z, which plays the same role as Z_s in the case of the genuine plane waves from the previous section, is

$$Z = a\exp\left[ik\Theta - \int^{\tau} d\tau\left\{is(t\eta)+\frac{1}{2}\,\Theta\right\}\right]\cdot e^{ins}. \tag{9.4}$$

Ignoring the terms denoted by $O(1/ik)$ in (9.1), we have

$$\psi_{A_1\ldots A_{2s}} = Zt_{A_1}\ldots t_{A_{2s}}. \tag{9.5}$$

Now, representing Z as $R\exp[iA]$, one can construct the planes of polarization $\Pi_{(r)E}$ and $\Pi_{(r)H}$ according to the scheme of the previous section. These planes, however, when one moves along the light ray from x^α to $x^\alpha+t^\alpha d\tau$, rotate around t^α through the angle

$$d\Omega = \frac{1}{s}\,\nabla_t A d\tau, \tag{9.6}$$

(compare (8.13)). Then, using (9.4) and $\nabla_t\Phi = 0$ one easily finds that

$$\frac{d\Omega}{d\tau} = -(t\eta) \tag{9.7}$$

independent of the spin s. This rotation of the planes of polarization when one travels with light rays (*i.e.* the rotation per unit of the affine parameter) forms a pure "curvature effect". Indeed, combining (4.16) and (4.17) with (9.7), we obtain

$$\frac{d\Omega}{d\tau} = \frac{2}{\sigma}\,\mathrm{Im}\,(W_{ABCD}t^A t^B t^C t^D). \tag{9.8}$$

It is sensible to talk about this effect only when $\sigma\neq 0$, so that the phase of t_A is determined by the canonical form of $\Phi_{;\alpha\beta}$. This effect is certainly absent when V_4 is flat or conformally flat. It also does not occur when $t_\alpha = \Phi_{;\alpha}$ coincides with one of Debever's vectors. Moreover, according to the consistency conditions (7.13) and (7.17), it also cannot appear for higher spins, $s>1$.

Therefore, the rotation (9.8) — a rotation of the planes of polarization due to curvature — can be understood as a sensible physical phenomenon only in the case of neutrinos ($s = 1/2$) and electromagnetic waves ($s = 1$) traveling in a V_4 with non-zero conformal curvature. In order to have this effect $t_\alpha = \Phi_{;\alpha}$ and $\Phi_{;\alpha\beta}$, which determine the phase of t_A, must be related to Weyl's tensor $W_{\alpha\beta\gamma\delta}$ so that Im $(W_{ABCD}t^At^Bt^Ct^D) \neq 0$. This, we repeat, prevents gravitational vectors (Debever's vectors) from having the orientations of t_α. Of course, the smaller σ is, the larger $\frac{d\Omega}{d\tau}$ is, but when $\sigma \to 0$ the phase of t_A is undetermined. In other words: The smaller σ is the more difficult the detection of $d\Omega/d\tau$ becomes.

It is clear that the formula (9.8) answers invariantly and for arbitary spins the problem studied in [1] and [2] in the electromagnetic case and in the approximation of weak gravitational field.

Finishing the discussion of the leading term of the optical development (9.1), we would like to point out that the "divergence" or "magnification" Θ affects the amplitude $\psi_0^{(0)}$ in the form of the factor exp $[-1/2\int^\tau \Theta d\tau]$. This was to be expected. The positive magnification $\Theta > 0$, *i.e.*, a divergent beam of "light rays" causes the amplitude of the waves (the "density" of waves) to decrease with increasing affine parameter. In the case of negative magnification, a convergent beam of "light rays", the amplitude increases with increasing affine parameter.

"Vectorial optics" also gives appreciable information about the higher terms of the "optical development". Although a detailed study of the structure of these "higher terms" lies outside of the scope of this paper, information which we have already gathered is sufficient to drow some general conclusions about the algebraic properties of the terms considered. There are two situations drastically distinguished: 1) $\sigma = 0$; 2) $\sigma \neq 0$.

When $\sigma = 0$ but $[\nabla_{\overline{m}} + i(\overline{m}\eta)s]\psi_0^{(0)} \neq 0$, then we have $\psi_1^{(1)} \neq 0$ (see (7.6a)) and $\psi_{(l+1)}^{(1)} = 0$ for $l \geqslant 1$ in the first order in $1/ik$. The amplitude $\psi_0^{(1)}$ may be non-zero or in the special case could vanish; it certainly must be non-zero when the "sources" of equation (7.8) are non--zero. Therefore, the terms of the first order have the form $\psi^{(1)}_{A_1\ldots A_{2s}} = \psi_0^{(1)}q^0_{A_1\ldots A_{2s}} + \psi_1^{(1)}q^1_{A_1\ldots A_{2s}}$. This makes clear that, while $\psi^{(0)}_{A_1\ldots A_{2s}}$ can be represented as $\alpha_{A_1}\ldots\alpha_{A_{2s}}$, the quantity $\psi^{(1)}_{A_1\ldots A_{2s}}$ has the form of $\alpha_{(A_1}\ldots\alpha_{A_{2s-1}}\beta_{A_{2s})}$. From the structure of the recurrence relations (7.1) one can easily see that the higher terms also exhibit a similar regularity: The term $(ik)^{-n}\psi^{(n)}_{A_1\ldots A_{2s}}$ is the product of $\alpha_{A_1}\ldots\alpha_{A_{2s-n}}$ times $\beta_{A_{2s-n+1}}\gamma_{A_{2s-n+2}}\ldots\xi_{A_{2s}}$ symmetrized over spinorial indices. The spinor α_A is form the leading term. The spinors $\beta_A, \gamma_A, \ldots \xi_A$ (n of them) are in general all different, but in special cases some of them may coincide. Of course, this statement makes sense for $2s - n \geqslant 0$.

A similar situation arises when $\sigma \neq 0$. One easily sees from the results of Section VII that, while $\psi^{(0)}{}_{A_1\ldots A_{2s}}$ has the structure $\alpha_{A_1}\ldots\alpha_{A_{2s}}$, the quantity $\psi^{(1)}{}_{A_1\ldots A_{2s}} = \sum_{l=0}^{2}\psi_l^{(1)}q_{A_1\ldots A_{2s}}$ has the algebraic structure of $\alpha_{(A_1}\ldots\alpha_{A_{2s-2}}\beta_{A_{2s-1}}\gamma_{A_{2s})}$. Two new spinors β_A and γ_A appear in the first order. From the recurrence relations (7.1) one can deduce that this is a general regularity. The spinor α_A is contained $2s-2n$ times in $\psi_{\,A_1\ldots A_{2s}}$ together with some $2n$ other

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spinors $\beta_A, \dots \xi_A$ in the form of an external product. Of course, this is totally symmetrized over the $2s$ spinorial indices. One can talk sensibly about this regularity of algebraic structure for $2s-2n \geq 0$.

Summing up this discussion, our "optical development" has the algebraic structure

$$\sigma = 0 \rightarrow \psi_{A_1 \dots A_{2s}} = \Big\{ \alpha_{A_1} \dots \alpha_{A_{2s}} + \frac{1}{ik} \alpha_{(A_1} \alpha_{A_{2s-1}} \beta_{A_{2s})} + $$
$$+ \frac{1}{(ik)} \alpha_{(A_1} \dots \alpha_{A_{2s-2}} \beta_{A_{2s-1}} \gamma_{A_{2s})} + \dots \Big\} \exp \{ik\,\Phi\} \tag{9.9a}$$

$$\sigma \neq 0 \rightarrow \psi_{A_1} \dots {}_{A_{2s}} = \Big\{ \alpha_{A_1} \dots \alpha_{A_{2s}} + \frac{1}{ik} \alpha_{(A_1} \dots \alpha_{A_{2s-2}} \beta_{A_{2s-1}} \gamma_{A_{2s}} +$$
$$+ \frac{1}{(ik)^2} \alpha_{(A_1} \dots \alpha_{A_{2s-4}} \beta_{A_{2s-3}} \gamma_{A_{2s-2}} \delta_{A_{2s-1}} \varepsilon_{A_{2s})} + \dots \Big\} \exp \{ik\Phi\}. \tag{9.9b}$$

In particular, for $s = 2$, using the standard graphical symbols for the Petrov-Penrose types of the $W_{ABCD} = \psi_{ABCD}$, we have

$$\sigma = 0 \rightarrow \psi_{A_1 \dots A_4} = \Big\{ \Vert\Vert + \frac{1}{ik} \,|||\, + \frac{1}{(ik)^2} \Big\rangle\Big\langle + \frac{1}{(ik)^3} \Big\rangle\Big\langle + \dots \Big\} \exp \{ik\Phi\} \tag{9.10a}$$

$$\sigma \neq 0 \rightarrow \psi_{A_1 \dots A_4} = \Big\{ \Vert\Vert + \frac{1}{ik} \Big\rangle\Big\langle + \frac{1}{(ik)^2} \Big\rangle\Big\langle + \dots \Big\} \exp \{ik\Phi\}. \tag{9.10b}$$

The optical "peeling off" property described by (9.9) shows some affinity to Sachs's development of the curvature along light rays [13] and the "peeling off" effects studied by Penrose [7] for the fields with spin s which are essentially the same as the fields studied in the present paper. But in the quoted works, however, spinorial objects were studied along light rays and their asymptotic behaviour was investigated; $1/\tau$ served as the parameter of the development.

The "peeling off" of the type (9.9) is something very different; here, it is $1/ik$ which serves as the parameter of the development.

Now, we would like to comment about the nature of the consistency conditions (7.13) and (7.17) derived on the level of "vectorial optics" and necessary if $s < 1$. These conditions impose some restrictions on the propagation vector t_α as related to the conformal curvature.

One can easily see that similar conditions are necessary quite generally, *i.e.*, without any approximations.

Consider the Ricci formula

$$\psi_{B_1 \dots B_{2s};\alpha;\beta} - \psi_{B_1 \dots B_{2s};\beta;\alpha} = \sum_{i=1}^{2s} R^{C_i}_{B_i \alpha\beta} \psi_{B_1 \dots C_i \dots B_{2s}}. \tag{9.11}$$

Contracting it with $g^{\alpha\dot{P}B_1} g^{\beta\dot{Q}B_2}$ and applying field equations $\nabla^{\dot{A}B_1} \psi_{B_1 \dots B_{2s}} = 0$, one derives the necessary algebraic condition

$$\sum_{i=1}^{2s} R^{C_i}_{B_i \alpha\beta} g^{\alpha\dot{P}B_1} g^{\beta\dot{Q}B_2} \psi_{B_1 \dots C_i \dots B_{2s}} = 0. \tag{9.12}$$

For $s \leq 1$ this is trivially fulfilled. For $s \geq 3/2$ by using (4.19), one proves that (9.12) is equivalent to

$$W_{A_1A_2A_3(B_3} \psi^{A_1A_2A_3}{}_{B_4 \dots B_{2s})} = 0. \tag{9.13}$$

This algebraic restriction on the solutions of the field equations is very strong. It is of interest to find its implications with W_{ABCD}'s of the 6 possible algebraic types of Petrov and Penrose. However, we will not elaborate this point in the present paper.

Substituting our optical development into (9.13) canceling by exp $(ik\Phi)$ and demanding the validity of (9.13) in all orders in $1/ik$ separately, one obtains

$$W_{A_1A_2A_3(B_3} \psi^{(n)A_1A_2A_3}{}_{B_4 \dots B_{2s})} = 0. \tag{9.14}$$

This condition for $n = 0$ can easily be seen to be equivalent to (7.13) and (7.17).

A few words more about the case of $s = 2$, the case of the gravitational radiation.

In a general V_4 with $R_{\alpha\beta} \neq 0$, *i.e.*, with some tensor of matter present, the Bianchi identities in the spinorial transcription are

$$R_{\alpha\beta[\gamma\delta;\varepsilon]} = 0 \leftrightarrow \begin{cases} \nabla^{\dot{A}B_1} W_{B_1B_2B_3B_4} = \dfrac{1}{8} \nabla_{\dot{S}(B_2} \nabla_{B_3B^4)}{}^{\dot{A}\dot{S}} & \text{(9.15a)} \\[2ex] \nabla^{\dot{R}S} U_{\dot{R}\dot{A}SB} - \dfrac{1}{4} \nabla_{\dot{A}B} R = 0, & \text{(9.15b)} \end{cases}$$

where the object $U_{\dot{A}\dot{B}CD} = \frac{1}{2} g_{\alpha\dot{A}C} g_{\beta\dot{B}D} [R^{\alpha\beta} - \frac{1}{4} g^{\alpha\beta} R]$ due to Einstein's equations $G_{\alpha\beta} = = \dfrac{8\pi\varkappa}{C^2} T_{\alpha\beta}$ may be understand as determined by the tensor of matter $T_{\alpha\beta}$ according to

$$U_{\dot{A}\dot{B}CD} = \frac{4\pi\varkappa}{C^2} g_{\alpha\dot{A}C} g_{\beta\dot{B}D} \left\{ \frac{1}{4} g^{\alpha\beta} g_{\nu\mu} T^{\nu\mu} - T^{\alpha\beta} \right\}. \tag{9.16}$$

The relations (9.15b) are equivalent to $G^{\alpha\beta}{}_{;\beta} = 0$ which implies $T^{\alpha\beta}{}_{;\beta} = 0$, *i.e.*, "equations of motion" of matter.

Now, (9.15a) can be understood as the field equations of the field of the gravitational radiation, W_{ABCD}, with the sources generated by the gradient of the tensor of matter, $T_{\alpha\beta;\gamma}$. The right hand member of (9.15a) can be interpreted as the "gravitational current". We have here the perfect analogy with the electromagnetic case. Maxwell's equations with currents, written in the spinorial transcription are $\nabla^{\dot{A}B} f_{BC} = \dfrac{\pi}{C} j^{\dot{A}}{}_C$, where $j_{\dot{A}B} = g_{\alpha\dot{A}B} j^\alpha$ is a hermitian object.

In the regions of space-time where $T_{\alpha\beta} = 0$, (9.15a) reduces to $\nabla^{\dot{A}B} W_{BCDE} = 0$, *i.e.*, to the equations studied in this paper. Notice that it follows from its derivation that (9.14)

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if is applyied for $s = 2$ it should hold for $\psi_{A_1\ldots A_4}$ only in the absence of gravitational currents. One finds it to be trivially satisfyied because $W_{A_1A_2A_3P}W^{A_1A_2A_3}{}_Q = \frac{1}{2}g_{PQ}W_{A_1\ldots A}W^{A_1\ldots A}$.

Solutions of Einsteins empty-space equations are certainly of interest, especially those which exhibit on the level of W_{ABCD} a wave-like properties. It is, however, the authors opinion that one can learn more about the physics of the gravitational radiation by studying $W_{A\dot{B}CD}$ as generated by the gradient of the energy-momentum tensor according to (9.15a). Notice that with the help of (9.15a) one can show that the effect of

$$\Box \underset{\mathrm{df}}{=} -g^{\nu\mu}\nabla_\nu\nabla_\mu \text{ acting on } W_{ABCD} \text{ is}$$

$$\left(\Box + \frac{1}{2}R\right)W_{ABCD} - 24W_{(AB}{}^{RS}W_{CD)RS} = -\frac{1}{8}\nabla^{\dot{R}}{}_{(A}\nabla^{\dot{S}}{}_{B}U_{CC)\dot{R}\dot{S}}, \tag{9.17}$$

wheere the sources are determined by second derivatives of the tensor of matter. Equations (9.15a) and (9.17) form a convenient starting point to study the emission of gravitational radiation by matter in motion. This can be done effectively at least in the weak field approximation.

As far as the difficult problem of the detection of the gravitational radiation is concerned, it seems that formula (9.8) can be of some importance at least theoretically. Namely one can study the polarization of electromagnetic waves as effected by the conformal curvature. Combining (9.8) with (4.9c) one has

$$4W_{ABCD}t^At^Bt^Ct^D = \nabla_t\sigma + \Theta\cdot\sigma + 2i\sigma\nabla_t\Omega. \tag{9.18}$$

The quantities t_A, σ, Θ, and Ω can be understood as related to a beam of electromagnetic waves. The quantities appearing on the right hand side of (9.18) are all in principle measurable. That way the left hand member of (9.18) referring to the conformal curvature, *i.e.*, ''gravitational radiation'', is in principle a measurable quantity.

It would be of interest to study (9.18) in the weak field approximation. The methodics of [2] may be applyied where the "scalar optics" was developed in the case of the weak, but otherwise entirely general, gravitational field. The weak gravitational field can be considered as given explicitly through the tensor of matter; there are available well-developped approximations procedures [22], [23] which give explicite expressions for the weak gravitational field as pressed through the tensor of matter. It will present no fundamental difficulty to compute σ, Θ, $\nabla_t\sigma$, and $\nabla_t\Omega$ due to the scattering of electromagnetic waves by a weak gravitation field.

It is the authors pleasure to express the gratitude to Dr Arturo Rosenblueth for the warm hospitality extended to him during his stay at the "Centro de Investigación y de Estudios Avanzados del Instituto Politecnico Nacional". The author is also grateful to Mr Bogdan Mielnik for helpfull discussions.

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