EQUATIONS OF MOTION OF A RELATIVISTIC BI-POINT MODEL OF FREE ELEMENTARY PARTICLE IN AN EXTERNAL GRAVITATIONAL FIELD

By Z. Borelowski

Department of Theoretical Physics, Jagellonian University, Cracow*

(Received April 4, 1965)

We consider a particle characterized by its rest mass m_0, coordinates x^α and a vector r^α related to the coordinates, assuming it to be space-like of constant spatial length l in the own reference system of the particle. Motion in Minkowski space is assumed to be determined by the variational principle $\delta\int_{\tau_1}^{\tau_2} L d\tau = 0$, where τ is the proper time of the particle and the Lagrangian L is given by $L = -m_0 c(\sqrt{u_\lambda u^\lambda} + \sqrt{-\dot{r}_\lambda \dot{r}^\lambda}) + \Lambda r_\lambda r^\lambda$. Moreover, c is the velocity of light, u^λ — the four-velocity, $\dot{r}_\lambda \overset{\mathrm{df}}{=} \frac{dr_\lambda}{d\tau}$, and Λ — Lagrange's factor to be determined from the equations of motion. The particle is found at be a spin particle. We denote the spin bi-vector by $s^{\alpha\beta}(s^{\alpha\beta} \sim r^\alpha \dot{r}^\beta - r^\beta \dot{r}^\alpha)$. The constant $l(r_\lambda r^\lambda = -l^2)$ is a measure of the particle's diffluence in its own reference system. Considering it as a test particle, we search for its equations of motion in an external gravitational field with potentials $g_{\mu\nu}$. In a first approach, we start from the same principle as in Minkowski space. Albeit now the Lagrangian is expressed by the formula $L = -m_0 c(\sqrt{g_{\lambda\nu} u^\lambda u^\nu} + \sqrt{-g_{\lambda\nu}\dot{r}^\lambda \dot{r}^\nu}) + \Lambda g_{\lambda\nu} r^\lambda r^\nu$, with $\dot{r}^\lambda \overset{\mathrm{df}}{=} u^\nu \nabla_\nu r^\lambda$ (∇_ν is the covariant derivative). In a second approach we recur to ideal liquid representation, starting from the well-known principle of least action, assuming traditionally action of the field as $S_f = \int R\sqrt{-g}\, d^4x$ and action of matter as $S_{\mathrm{int}} = \int L\sqrt{-g}\, d^4x$. The equations of motion obtained by the two methods are identical. It results that L is a first integral of the equations of motion, the module of spin $(s^{\lambda\nu}s_{\lambda\nu})$ is constant in the time, and the gravitational field only modifies the shape of the equations to functions $u^\alpha = u^\alpha(\tau)$ (thus modifying motion of the particle as a whole) but does not modify the shape of the equations of motion from which we obtain the dependence $r^\alpha = r^\alpha(\tau)$; this vector and its derivative with respect to the own time generate the bi-vector of spin of our particle.

1. *Introduction*

In our previous paper [1] on the "Equations of Motion of a One-Point Model of Elementary Particle in an External Gravitational Field" we proposed a procedure for deriving the equations of motion of a test free spin particle in an external gravitational field. That

* Address: Instytut Fizyki U J Kraków, ul. Reymonta 4, Polska.

was a particle defined by motion of a point singularity, whose Lagrangian in Minkowski space was of the form $L_1 = -m_0 c\left(\sqrt{u_\lambda u^\lambda} - \frac{l^2}{c^3} w_\lambda w^\lambda\right)$, with m_0 denoting the rest mass, u^λ — the four-velocity, w^λ — the four-acceleration, c — the velocity of light and l — a characteristic constant having the dimension of a length, which (as resulted from the equations of motion) was a measure of the diffluence of our particle[1] [2].

Since L_1 depends on the w^α, the four-vector of energy-momentum p^α of the particle is not in general parallel to its four-velocity and consequently we have a spin-particle. From the equations of motion of our singularity a relation was obtained between the rest mass of the particle, its spin, and the constant l the interpretation of which as applied to the case of barion resonance permits to obtain a relationship between the mass (energy) and spin not greatly divergent from the experimental data. Exactly the same result can be obtained by the construction of the relativistic two-point model of free elementary particle proposed in [3]. That model was evolved in connection with certain concepts of bi-local field theory [4]. The present paper is concerned with the derivation of equations of motion for precisely that bi-point model of elementary particle in an external gravitational field. Herein, the particle represented by the model is dealt with as a test particle *i.e.* its own gravitational field is neglected. In particular, we shall be concerned with the following problems:

1) What will be the procedure to be followed when applying consequently the formalism of general relativity for generalizing the variational principles defining motion of test particles in Minkowski space to a world with curvature?

2) In what form will the effect of the external gravitational field become apparent in the equations of motion of our particle?

It should be stressed, however, that (as in deriving the respective equations of motion for the one-point model [1]) any application of the formalism of a patently macroscopic theory such as general relativity to typically and essentially microscopic models of elementary particles can claim to be no more than an attempt to assess qualitatively the results thus derived.

2. *Short discussion of the bi-point model of elementary particle in Minkowski space*

Let us consider in Minkowski space (with metric $-, -, -, +$) a free system of two point singularities with relative coordinates r^α and "centre of mass" coordinates x^α. We assume motion of this system (bi-point) to be given by the principle of least action

$$\delta \int_{\pi_1}^{\pi_2} L(r^\lambda, \dot{r}^\lambda, u^\lambda)\, d\pi = 0, \tag{2.1}$$

[1] Einstein's summation convention is applied. Greek indices take the values 1, 2, 3, 4, Latin indices — 1, 2, 3. A dot denotes the derivative with respect to the own time $\left(e.g.\ \dot{a}^\lambda = \frac{da^\lambda}{d\tau}\right)$ and from Section 3 onward (*i.e.* when recurring to tensor calculus in Riemann space) — the absolute derivative with respect to the own time taken along the trajectory of the particle (*e. g.* $\dot{b}_\lambda = u^\nu \nabla_\nu b_\lambda$, with ∇_ν standing for the covariant derivative).

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

wherein

$$\delta x^\alpha(\pi_1) = \delta x^\alpha(\pi_2) = \delta r^\alpha(\pi_1) = \delta r^\alpha(\pi_2) = 0, \tag{2.2}$$

with π denoting an as yet arbitrary parameter, $u^\alpha \overset{\text{df}}{=} \frac{dx^\alpha}{d\tau}$, x^α and r^α considered as independent, and δ — symbolizing variation of the functions x^α (π) and r^α (π) without variation of π. The postulate that the action $S = \int_{\pi_1}^{\pi_2} L(r^\alpha, \dot r^\alpha, u^\alpha) d\pi$ shall be invariant with respect to a change in parametrization $\pi = \pi\,(\bar\pi)$ (where $\bar\pi$ is a new, also arbitrary parameter) leads [3] to L in the (simplest) form

$$L = -m_0 c \left(\sqrt{u_\lambda u^\lambda - \frac{(u_\lambda r^\lambda)^2}{r_\lambda r^\lambda}} + \sqrt{-\dot r^\alpha \dot r_\alpha + \frac{(r_\alpha \dot r^\alpha)^2}{r_\alpha r^\alpha}} \right),$$

$$\left(\dot r^\alpha \overset{\text{df}}{=} \frac{dr^\alpha}{d\pi} \right). \tag{2.3}$$

Introducing the canonical momenta we split them in two groups: the "external momenta" p^α and "internal momenta" k^α as follows:

$$p_\lambda \overset{\text{df}}{=} \frac{\partial L}{\partial u^\lambda}, \tag{2.4}$$

$$k_\lambda \overset{\text{df}}{=} \frac{\partial L}{\partial \dot r^\lambda}. \tag{2.5}$$

Eqs (2.4), (2.5) and (2.3) now yield

$$p_\lambda p^\lambda = -k_\lambda k^\lambda = m_0^2\, c^2 \tag{2.6}$$

whence p_λ is seen to be a time-like and k_λ a space-like four-vector. From (2.4) and (2.3) we have moreover

$$r_\lambda p^\lambda = 0 \tag{2.7}$$

and r^λ is space-like in an inertial reference system wherein $p^i = 0$ and r^4 vanishes also. On the assumption that in the reference system considered the spatial distance between the point singularities is constant amounting to l we obtain from what has been said an additional condition which, subsequent to isoperimetrization, has to be taken into account in deriving equations of motion from (2.1). This condition is of the form

$$r_\lambda r^\lambda = \text{const} \overset{\text{df}}{=} -l^2 \tag{2.8}$$

Eqs (2.8), (2.7), (2.4), (2.6) with (2.3) yield $r_\lambda \dot r^\lambda = 0$
and

$$u_\lambda r^\lambda = 0 \tag{2.9}$$

and the total Lagrangian L_c *i.e.* on considering the condition (2.8) is of the form

$$L_c = -m_0 c\,(\sqrt{u_\lambda u^\lambda} + \sqrt{-\dot{r}_\lambda \dot{r}^\lambda}) + \Lambda\, r_\lambda r^\lambda, \tag{2.10}$$

where Λ is Lagrange's factor to be determined from the equations of motion. Putting $\pi = \tau$ where τ is the proper time of the centre of our bi-point, we obtain from the variational principle (2.1), (2.2) with L replaced by L_c of Eq. (2.10) equations of motion which, in the reference system Σ_0 (*i.e.* wherein $p^i = 0$), can be written thus:

$$w^\alpha = 0, \tag{2.11}$$

$$\ddot{r}^\alpha = -\frac{\dot{r}_\lambda \dot{r}^\lambda}{r_\eta r^\eta}\, r^\alpha. \tag{2.12}$$

From the above equations it results that the singularities move along a circumference of radius l. The constant l provides a measure of the diffluence of the particle at rest in Σ_0 ($u^\alpha \overset{\Sigma_0}{=} 0$) as represented by the system of both singularities as a whole together with their path in the reference system.

It is found that the total torque of the bi-point (*cf.* [3]) is given by the formula

$$K^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha + s^{\alpha\beta}, \tag{2.13}$$

where

$$s^{\alpha\beta} \overset{\mathrm{df}}{=} r^\alpha k^\beta - r^\beta k^\alpha \tag{2.14}$$

so that $s^{\alpha\beta}$ represents its spin.

From the equations of motion, it results that in Σ_0 the spin s_{Σ_0} of the particle is constant and is given as

$$s_{\Sigma_0} = \sqrt{s^{\alpha\beta} s_{\alpha\beta}} = \sqrt{2} m_0 c l = \text{const.} \tag{2.15}$$

This is precisely the relation between the spin and mass of the particle to which reference was made in the Introduction.

Incidentally, we have nowhere recurred to the fact that r^α is a difference of two four-vectors. Thus, we can state the problem otherwise, by requiring that the new procedure shall yield the same results as the old procedure but that it shall be better adapted for generalizations to Riemann space. Namely, we assume that a physical reality (of the type of an elementary spin particle) is represented in Minkowski space by a point-moment x^α and space-like four-vector r^α of constant space-length l in the reference system Σ_0 (*i.e.* in the system wherein the space-momentum of the particle wanishes). Its motion is defined by the variational principle

$$\delta \int_{\pi_1}^{\pi_2} L d\pi = 0, \tag{2.16}$$

with the condition (2.2) Its spin is obviously that of Eq. (2.14) and all results derived hitherto remain the same.

3. Derivation of the equations of motion of the our "bi-point" in an external gravitational field

In deriving the equations of motion we shall use (as when deriving those of the point singularity whose Lagrangian depended on higher derivatives of x^α[1]) two procedures. The one is a simple generalization of the principle (2.16), (2.2) in Riemann space and yields the equations of motion, whereas the other is closer to the field approach and yields simultaneously the field equations and those of motion. We shall now state in brief what they consist in.

Let us consider Riemann space with metric tensor $g_{\mu\nu}$ and a particle whose motion in plane space is determined by the variational principle $\delta\int\limits_{\tau_1}^{\tau_2} L d\tau = 0$. The Lagrangian is a scalar function of certain four-vectors a^α and b^α which are functions of the proper time of the particle along its trajectory, which we are searching for. Its equations of motion in a gravitational field of potential $g_{\mu\nu}$ are found as follows: we replace in the Lagrangian the products $a_\lambda b^\lambda$, $a_\lambda a^\lambda$, $b_\lambda b^\lambda$ by $g_{\lambda\nu}a^\lambda b^\nu$, $g_{\lambda\nu}a^\lambda a^\nu$, $g_{\lambda\nu}b^\lambda b^\nu$ and consider a set of competing trajectories with points P_1 and P_2 in common corresponding to the values τ_1 and τ_2 of the own time. Motion of the particle is now determined by the variational principle

$$\delta \int\limits_{\tau_1}^{\tau_2} L(g_{\lambda\eta}a^\lambda b^\eta,\, g_{\lambda\eta}a^\lambda a^\eta,\, g_{\lambda\eta}b^\lambda b^\eta)\, d\tau = 0. \tag{3.1}$$

so that the equations of motion are given by the Euler-Lagrange equations resulting from the above principle.

In the other approach, we start from the principle of least action $\delta(S_f + kS_{\text{int}}) = 0$. Here, $S_f \stackrel{\text{df}}{=} \int R\sqrt{-g}\, d^4x$, where R is the tensor of curvature and $g \stackrel{\text{df}}{=} \text{Det}\,(g_{\mu\nu}) < 0$. $S_{\text{int}} = \int L\sqrt{-g}\, d^4x$. S_f is the action of the field alone, and S_{int} — that of the particle interacting with the field. The symbol δ stands for variation of the metric tensor as well as for variation of the four-vectors a^λ and b^λ (involved in L) due to an infinitesimal transformation of the coordinates x^α. Hence, by the principle of least action we have

$$\int (G^{\mu\nu} - kT^{\mu\nu})\, \delta g_{\mu\nu} \sqrt{-g}\, d^4x + \int \frac{\delta L}{\delta a^\lambda}\, \delta a^\lambda \sqrt{-g}\, d^4x + $$
$$+ \int \frac{\delta L}{\delta b^\lambda}\, \delta b^\lambda \sqrt{-g}\, d^4x = 0 \tag{3.2}$$

with the notation:

$\dfrac{\delta}{\delta a^\lambda}$ — variational derivative with respect to a^λ and similarly with respect to b^λ,

$G^{\mu\nu}$ — Einstein's tensor,

$T^{\mu\nu}$ — the tensor of momentum-energy of the particle $\left(T^{\mu\nu} = -2\dfrac{\delta L}{\delta g_{\mu\nu}}\right)$,

k — the well-known constant involving the constant of gravitation equalizing the dimensions of the two preceding tensors.

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

Hence, we obtain simultaneously the equations of motion:

$$\frac{\delta L}{\delta a^{\alpha}} = 0,$$

$$\frac{\delta L}{\delta b^{\alpha}} = 0, \tag{3.3}$$

and the field equations

$$G^{\mu\nu} = kT^{\mu\nu}, \tag{3.4}$$

the latter resulting from the former, but not necessarily inversely.

We now proceed to derive in detail the equations of motion of our bi-point, whose Lagrangian in Minkowski space is given by (2.10). Indeed, one can readily show that on the assumption that it is given by (2.3) and on taking into account the assumption (2.8), the relation of Eq. (2.9) is fulfilled in Riemann space also.

A. First approach

We consider in Riemann space with metric tensor $g_{\mu\nu}$ a particle characterized by a point singularity with coordinates x^{α} and space-like four-vector r^{α} related to the singularity. We assume motion of the singularity to be detemined by the variational principle

$$\delta \int\limits_{\tau_1}^{\tau_2} L d\tau = 0, \tag{3.5}$$

the Lagrangian L being given by the formula

$$L = -m_0 c\left(\sqrt{g_{\mu\nu} u^{\mu} u^{\nu}} + \sqrt{-g_{\lambda\eta} \dot{r}^{\lambda} \dot{r}^{\eta}}\right) + \Lambda g_{\mu\nu} r^{\mu} r^{\nu}, \tag{3.6}$$

with the notation:

m_0 — rest mass of the particle,
c — velocity of light,
$u^{\nu} \overset{\mathrm{df}}{=} \frac{dx^{\nu}}{d\tau}$ — four-velocity of the particle,
$\dot{r}^{\nu} \overset{\mathrm{df}}{=} u^{\lambda} \nabla_{\lambda} r^{\nu}$, where r^{ν} is the four-vector characterizing the particle,
Λ — a Lagrange factor to be determined from the equations of motion, and
τ — the proper time of the particle.

The four-vector r^{α} is a function of the x^{α}. Since motion of the particle is given by the set of functions $x^{\alpha} = x^{\alpha}(\tau)$, we shall consider in this approach the components r^{λ} as functions of the own time of the particle:

$$r^{\lambda} = f^{\lambda}(x^{\beta}(\tau)) = r^{\lambda}(\tau). \tag{3.7}$$

With regard to this, we supplement the principle (3.5) by the condition (2.2). Accordingly, the Euler-Lagrange equations resulting from (3.5), (2.2) as well as (3.7) are of the form

$$\frac{d}{d\tau} \frac{\partial L}{\partial u^{\alpha}} - \frac{\partial L}{\partial x^{\alpha}} = 0, \tag{3.8}$$

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} = 0, \tag{3.9}$$

$$\left(\frac{\partial L}{\partial \frac{dr^\alpha}{d\tau}} = \frac{\partial L}{\partial \dot{r}^\alpha}\right).$$

On introducing the canonical momenta

$$p_\lambda \overset{\text{df}}{=} \frac{\partial L}{\partial u^\lambda} \tag{3.10}$$

and

$$k_\lambda \overset{\text{df}}{=} \frac{\partial L}{d\dot{r}^\lambda} \tag{3.11}$$

(p_λ to be referred to as the external momentum, and k_λ — as the internal momentum) we rewrite the equations of motion as

$$\frac{d}{d\tau}p_\lambda = \frac{\partial L}{\partial x^\lambda}, \tag{3.8'}$$

$$\frac{d}{d\tau}k_\lambda = \frac{\partial L}{\partial r^\lambda}, \tag{3.9'}$$

or equivalently

$$\dot{p}_\lambda = \frac{\partial L}{\partial x^\lambda} - \Gamma^\nu_{\lambda\eta} p_\nu u^\eta, \tag{3.8''}$$

$$\dot{k}_\lambda = \frac{\partial L}{\partial r^\lambda} - \Gamma^\nu_{\lambda\eta} k_\nu u^\eta, \tag{3.9''}$$

where $\Gamma^\nu_{\lambda\eta}$ are Christoffel's symbols of the second kind, and $\dot{p}_\lambda$, $\dot{k}_\lambda$—the absolute derivatives with respect to the own time of the two four-momenta along the trajectory of the particle.

By (3.10), (3.11) and (3.6) we have

$$k_\lambda = \frac{m_0 c}{\sqrt{-g_{\nu\eta}\dot{r}^\nu \dot{r}^\eta}} \dot{r}_\lambda \tag{3.12}$$

and

$$p_\lambda = -m_0 u_\lambda + \frac{m_0 c}{\sqrt{-g_{\alpha\beta}\dot{r}^\alpha \dot{r}^\beta}} \dot{r}_\eta \nabla_\lambda r^\eta \tag{3.13}$$

∇_λ standing for the covariant derivative. We note that k_λ and p_λ are vectors and that k_λ is parallel to r_λ but p_λ is not parallel to u_λ, which is characteristic of spin particles. We first consider Eqs (3.9). On substituting into Eqs (3.9'') k_λ from (3.12) and L from (3.6), we

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

obtain after some computations the following form of Eqs (3.9):

$$\frac{\ddot{r}_\alpha}{\sqrt{-\dot{r}_\nu \dot{r}^\nu}} + \frac{(\dot{r}_\lambda \ddot{r}^\lambda)\,\dot{r}^\alpha}{(-\dot{r}_\eta \dot{r}^\eta)^{3/2}} - \frac{2\Lambda}{m_0 c}\, r_\alpha = 0 \tag{3.14}$$

(obviously, $\dot{r}_\nu \dot{r}^\nu = g_{\lambda\eta}\, \dot{r}^\lambda \dot{r}^\eta$). On multiplying both sides of (3.14) by r^α, effecting summation, and recurring to the fact that $r_\lambda r^\lambda = \text{const} = -l^2 i.e.\ r_\lambda \dot{r}^\lambda = 0$, we obtain

$$m_0 c\sqrt{-\dot{r}_\lambda \dot{r}^\lambda} = 2\Lambda\, r_\eta r^\eta = \text{const} \tag{3.15}$$

whence

$$\dot{r}_\lambda \dot{r}^\lambda = \text{const} \tag{3.16}$$

and

$$\Lambda = \frac{m_0 c\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}}{2 r_\eta r^\eta}. \tag{3.17}$$

Thus, the Lagrangian L is a first integral of the equations of motion (as in Minkowski space) and Eqs (3.9) with regard to (3.12), (3.16) and (3.17) become

$$\ddot{r}^\alpha = -\frac{\dot{r}_\lambda \dot{r}^\lambda}{r_\eta r^\eta}\, r_\alpha \tag{3.18}$$

thus being identical with Eqs (2.12). Here we have equations which in Minkowski space give the dependence of $r^\lambda = r^\lambda(\tau)$ since they do not involve the four-velocity and can thus be at once effectively integrated. From Eq. (3.18) it is seen that the external gravitational field does not affect the above equations.

The same equations can be derived in a somewhat different manner. Indeed, if we consider the components of the four-vector r^α as functions not of the time proper τ but rather of the coordinates x^α of the particle, we obtain instead of Eqs (3.9) the following equations:

$$\frac{d}{dx^\nu}\frac{\partial L}{\partial\left(\dfrac{\partial r^\alpha}{\partial x^\nu}\right)} - \frac{\partial L}{\partial r^\alpha} = 0 \tag{3.19}$$

which differ in form Eqs (3.9). However, on considering that $\dot{r}^\lambda = u^\nu \nabla_\nu r^\lambda = \dfrac{dr^\lambda}{d\tau} + \Gamma^\lambda_{\varepsilon\eta} r^\varepsilon u^\eta$, we obtain from (3.6)

$$\frac{d}{dx^\nu}\frac{\partial L}{\partial\left(\dfrac{\partial r^\alpha}{\partial x^\nu}\right)} = \dot{k}^\alpha + \Gamma^\lambda_{\alpha\eta} k_\lambda u^\eta \tag{3.20}$$

and

$$\frac{\partial L}{\partial r^\alpha} = \frac{m_0 c}{\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}}\ \dot{r}_\beta\, \Gamma^\beta_{\alpha\eta}\, u^\eta + 2\Lambda r_\alpha. \tag{3.21}$$

From the above relations and (3.12) we obtain that Eqs (3.19) take the form (3.18), as was to be expected.

We shall now proceed to write out in full the equations of motion (3.8). By (3.10), (3.8′) and (3.8″), they can be written as follows:

$$\dot{p}_\alpha + \Gamma^\lambda_{\alpha\eta} u^\eta p_\lambda - \frac{\partial L}{\partial x^\alpha} = 0. \tag{3.22}$$

Substituting herein L from Eq. (3.6) and p_α from (3.13), we obtain after some easy calculations the following form of the equations:

$$- m_0 w_\alpha + \frac{m_0 c}{\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}} (\ddot{r}_\lambda \nabla r^\lambda + \dot{r}_\eta u^\nu \nabla_\nu \nabla_\alpha r^\lambda - C_\alpha) -$$

$$- \frac{m_0 c}{2\sqrt{-\dot{r}_\nu \dot{r}^\nu}} \nabla_\alpha (\dot{r}_\eta \dot{r}^\eta) - \Lambda \nabla_\alpha (r_\lambda r^\lambda) = 0, \tag{3.22′}$$

where

$$C_\alpha \overset{\mathrm{df}}{=} -(\dot{r}_\lambda \nabla_\alpha u^\nu)(\nabla_\nu r^\lambda).$$

Now, $r_\lambda r^\lambda = -l^2 = \text{const}$, and since l being a measure of diffluence of the particle at rest is independent of the x^α,

$$\nabla_\alpha (r_\lambda r^\lambda) = -\nabla_\alpha l^2 = 0. \tag{3.23}$$

Hence and by (3.17), we have also

$$\nabla_\alpha (\dot{r}_\nu \dot{r}^\nu) = 0 \tag{3.24}$$

(indeed, by introduction of the constant Λ, $\nabla_\alpha \Lambda = 0$). Consequently, after some slight transformations and on recurring to (3.18), Eqs. (3.21) become

$$-m_0 w_\alpha + \frac{m_0 c}{\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}} \dot{r}_\eta u^\nu (\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) r^\eta = 0 \tag{3.25}$$

but

$$\dot{r}_\eta u^\nu (\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) r^\eta = - \dot{r}_\eta u^\nu R^\eta_{\varepsilon,\nu\alpha} r^\varepsilon = \frac{1}{2} R_{\alpha\nu,\lambda\eta}\, u^\nu (\dot{r}^\lambda r^\eta - \dot{r}^\eta r^\lambda), \tag{3.26}$$

where $R^\eta_{\varepsilon,\nu\alpha}$ is the tensor of curvature.

Thus if we define the bi-vector of spin $s^{\nu\alpha}$ of our particle in the same manner as without a gravitational field (the shape of the equations of motion determining the dependence of r^λ on τ remains unaffected by the gravitational field, and r^λ as well as $\dot{r}^\lambda$ enter the spin-bivector of the particle) *i.e.* in accordance with Eq. (2.14), then from (2.14), (3.12), (3.25) and (3.26) after easy calculations we obtain finally the equations of motion in the form

$$m_0 w_\alpha + \frac{1}{2} R_{\alpha\nu,\lambda\eta}\, s^{\lambda\eta} u^\nu = 0. \tag{3.27}$$

(From (2.16) and (3.14) it is seen that in the gravitational field, too, $s_{\alpha\beta}\, s^{\alpha\beta} = \text{const}$).

Thus, the effect of a gravitational field is seen to cause solely a change in the form of the equations of motion determining the dependence $u^\alpha = u^\alpha(\tau)$ *i.e.* defining motion of the particle as a whole. It leaves unaffected the shape of the "equations of motion" defining the dependence of the r^α on the x^β (*i.e.* of the r^α on τ) and consequently has no bearing on the diffluence of the particle and its spin. These are just the results one would be entitled to expect for the test particle.

B. Second approach

We shall now recur to a different method of deriving the equations of motion. For this purpose, we shall adapt the well-known method employed *e.g.* by Fok in his monograph [5] to our case of a bi-point particle. Let us consider the model of an ideal liquid of density $\mu = \mu(x^\alpha)$. We assume the continuity equation

$$\nabla_\nu(\mu u^\nu) = 0 \tag{3.28}$$

to be fulfilled ($u^\nu = u^\nu(x^\alpha)$ is now the velocity field, and r^λ — a vector field $r^\lambda = r^\lambda(x^\alpha)$).

We start from the principle of least action

$$\delta(S_f + kS_{\text{int}}) = 0, \tag{3.29}$$

where S_f is the action of the gravitational field alone,

$$S_f \overset{\text{df}}{=} \int R\sqrt{-g}\, d^4x \tag{3.30}$$

and S_{int} the "action of matter" interacting with it:

$$S_{\text{int}} \overset{\text{df}}{=} \int L\sqrt{-g}\, d^4x. \tag{3.31}$$

Here, R is the scalar of curvature, $g = \mathrm{Det}(g_{\nu\lambda}) < 0$, while L denotes the Lagrangian density as given by (3.6) with m_0 replaced by the function $\mu = \mu(x^\alpha)$ *i.e.* mass by mass density, and k is the well-known constant equalizing the dimensions of S_f and S_{int} (k contains the gravitation constant). The integral extends over the whole four-space (d^4x is the element of four-volume). Variation denoted by the symbol δ comprises both variation of $g_{\lambda\nu}$ and variation of the four-vectors u^α and r^α resulting from an infinitesimal transformation of the coordinates:

$$x'^\alpha = x^\alpha + \xi^\alpha. \tag{3.32}$$

It will be remembered that $\delta g_{\lambda\nu}$ is expressed as follows:

$$\delta g_{\lambda\nu} = \nabla_\lambda \xi_\nu + \nabla_\nu \xi_\lambda. \tag{3.33}$$

On introducing the Lagrange parameters l^α and their variations generated by the transformation (3.32) and expressed by the x^β, and denoting them by η^α, the variations δu^α as well-known from [5] are given by the η^α as follows:

$$\delta u^\alpha = u^\nu \nabla_\nu \eta^\alpha - \eta^\nu \nabla_\nu u^\alpha - \frac{1}{c^2}\, u^\alpha u^\nu (\nabla_\nu \eta^\lambda) u_\lambda. \tag{3.34}$$

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

On the other hand, it will be remembered that the variations δr^λ can be expressed as

$$\delta r^\lambda = \xi^\nu \nabla_\nu r^\lambda - r^\nu \nabla_\nu \xi^\lambda. \tag{3.35}$$

It the ξ^ν are arbitrary, the δr^λ are so too. We note that the postulate that the δr^λ shall vanish on the surface Σ encompassing the region of integration is equivalent to assuming that both the vector field ξ^ν and its first derivatives with respect to the x^α vanish on Σ.

The principle of least action (3.29), by (3.30), (3.31) and (3.34) can now be rewritten thus

$$\int \frac{\delta}{\delta g_{\eta\nu}} \sqrt{-g}\,(R+L)\,\delta g_{\mu\nu} d^4x + k \int \frac{\delta}{\delta u^\alpha} (\sqrt{-g}\,L) \left(\dot{\eta}^\alpha - \eta^\nu \nabla_\nu u^\alpha - \right.$$

$$\left. - \frac{1}{c^2} u^\alpha u^\nu (\nabla_\nu \eta^\lambda) u_\lambda \right) d^4x + k \int \frac{\delta}{\delta r^\alpha} (\sqrt{-g}\,L)\, \delta r^\alpha d^4x = 0. \tag{3.36}$$

We note that $g_{\lambda\eta}$ does not depend on the r^λ, whence $\frac{\delta g}{\delta r^\lambda} = 0$. Integrating the second term in (3.36) per partes, recurring to the continuity equation (3.28), and employing the Ostrogradsky-Gauss theorem as well as the assumption of η^λ vanishing on Σ (*i.e.* on the surface enclosing the region of integration), with the substitution of (3.6) for L, we have, after appropriate calculations (see also [5])

$$\int \sqrt{-g}(G^{\mu\nu} - kT^{\mu\nu})\,\delta g_{\mu\nu} d^4x + \int \sqrt{-g}\,Q_\alpha \eta^\alpha d^4x + \int \sqrt{-g}\,R_\alpha dr^\alpha d^4x = 0. \tag{3.37}$$

Here, $G^{\mu\nu}$ is Einstein's tensor, $T^{\mu\nu}$ — the tensor of momentum-energy $\left(T^{\mu\nu} = -2 \frac{\delta L}{\delta g_{\eta\nu}} \right)$,

$$Q_\alpha \overset{\mathrm{df}}{=} \frac{2}{3c} \left(1 + \frac{3}{2c} \sqrt{-\dot{r}_\lambda \dot{r}^\lambda} \right) w_\alpha - \dot{z}_\alpha - z_\lambda \nabla_\alpha u^\lambda + \frac{1}{c^2} u^\nu \nabla_\nu (u^\lambda z_\lambda u_\alpha), \tag{3.38}$$

with

$$z_\lambda \overset{\mathrm{df}}{=} \frac{\dot{r}_\nu \nabla_\lambda r^\nu}{\sqrt{-\dot{r}_\eta \dot{r}^\eta}}, \tag{3.39}$$

$$R_\alpha \overset{\mathrm{df}}{=} \frac{\ddot{r}_\alpha}{\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}} + \frac{\dot{r}_\lambda \ddot{r}^\lambda}{(-\dot{r}_\lambda \dot{r}^\lambda)^{3/2}} \dot{r}_\alpha - 2\Lambda r_\alpha. \tag{3.40}$$

The field equations are obtained on equating to zero the coefficients of $\delta g_{\mu\nu}$. We thus obtain

$$G^{\mu\nu} = kT^{\mu\nu} \tag{3.41}$$

and as $\Delta_\nu G^{\mu\nu} = 0$, we obtain the "dynamical equations"

$$\nabla_\nu T^{\mu\nu} = 0. \tag{3.41'}$$

The tensor $T^{\mu\nu}$ (as seen from (3.14)) is symmetric.

The equations of motion are obtained equating to zero turn in turn the coefficients Q_α and R_α of the mutually independent variations η^α and δr^α. We thus have

$$Q_\alpha = 0, \tag{3.42}$$

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

and simultaneously

$$R_\alpha = 0. \tag{3.43}$$

As seen from (3.37), the equations of motion imply the field equations; the inverse, however, is not generally true.

Multiplying both sides of (3.40) by r^α, we obtain (by (3.43))

$$\dot{r}_\lambda \dot{r}^\lambda = -4\Lambda^2 (r_\lambda r^\lambda)^2 = \text{const} \tag{3.44}$$

so that (by (3.44) and (3.40)) Eqs (3.43) assume the form:

$$\ddot{r}_\lambda = -\frac{\dot{r}_\nu \dot{r}^\nu}{r_\eta r^\eta} r_\lambda \tag{3.45}$$

which is identical with that of the respective equations obtained in our previous, first approach.

Multiplying now (3.42) by u^α and recurring to (3.39), we have

$$u_\lambda \dot{z}^\lambda = -\sqrt{-\dot{r}_\lambda \dot{r}^\lambda} = \text{const.} \tag{3.46}$$

From (3.38), (3.39), (3.45), (3.46) as well as by (3.23) and (3.24) it results that Eqs. (3.42) can be put in the form

$$w_\alpha + \frac{c}{\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}} \dot{r}_\nu u^\eta (\nabla_\eta \nabla_\alpha - \nabla_\alpha \nabla_\eta) r^\nu = 0. \tag{3.47}$$

This, however, is identical with Eq. (3.25). Consequently, on introducing the bi-vector of spin $s^{\nu\lambda}$ as previously we obtain

$$\mu w_\alpha + \frac{1}{2} R_{\alpha\nu,\lambda\eta} u^\nu s^{\lambda\eta} = 0. \tag{3.48}$$

Thus, the two methods lead to the same equations of motion, as could have been expected.

To conclude, we shall derive explicitly the rensor of momentum-energy of our particle. To this aim, we take the variation of the integral $\int \sqrt{-g}\, L\, d^4x$ applying the symbol $\underset{g}{\delta}$ to the gravitational potentials $g_{\lambda\nu}$ only. By (3.30) we have

$$\underset{g}{\delta} S_{\text{int}} = \underset{g}{\delta} \int \sqrt{-g}\, L d^4x = \int \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g}\, L\, \delta d_{\mu\nu} d^4x = -\frac{1}{2} \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g}\, dx^4. \tag{3.49}$$

Introducing herein L from (3.6) and after well-known and easy calculations (see, *e.g.* [5]), we obtain

$$\underset{g}{\delta} S_{\text{int}} = \int \left(\left(-\frac{1}{2} M u^\mu u^\nu + \frac{\mu c}{\lambda} \frac{\dot{r}^\nu \dot{r}^\mu}{\sqrt{-\dot{r}_\lambda \dot{r}^\lambda}} - \mu c \Lambda r^\mu r^\nu \right) \sqrt{-g}\, \delta g_{\mu\nu} + \right.$$
$$\left. + \frac{\mu c}{\sqrt{-\dot{r}_\nu \dot{r}^\nu}} \dot{r}^\lambda \underset{g}{\delta} \Gamma^\lambda_{\varepsilon\eta} u^\varepsilon r^\eta \sqrt{-g} \right) d^4x, \tag{3.50}$$

where M is a constant. However, obviously

$$\delta \Gamma^\lambda_{\nu\eta} = \frac{1}{2} g^{\lambda\beta} (\nabla_\eta \delta g_{\beta\nu} + \nabla_\beta \delta g_{\nu\eta} - \nabla_\beta \delta g_{\eta\nu}). \tag{3.51}$$

ACTA PHYSICA POLONICA Vol. XXVIII (1965) Fasc. 5

Substituting (3.51) in (3.50), integrating per partes, making use of the continuity equation and the Ostrogradsky-Gauss theorem and assuming that $\delta g_{\lambda\eta}$ vanishes on the surface bounding the integration region, we obtain after some computations that

$$T^{\mu\nu} = Mu^{\mu}u^{\nu} - \frac{\mu c}{\sqrt{-\dot{r}_{\lambda}\dot{r}^{\lambda}}}\,\dot{r}^{\mu}\dot{r}^{\nu} + 2\Lambda r^{\mu}r^{\nu} +$$

$$+ \frac{c}{\sqrt{-\dot{r}_{\lambda}\dot{r}^{\lambda}}}\,V^{\eta}\mu\left(r^{\eta}(u^{\nu}\dot{r}^{\mu} + u^{\mu}\dot{r}^{\nu}) - \frac{1}{2}\,\dot{r}^{\eta}\,(u^{\nu}r^{\mu} + u^{\mu}r^{\nu})\right), \tag{3.52}$$

with

$$M \overset{\mathrm{df}}{=} \mu\left(1 + \frac{\sqrt{-\dot{r}_{\lambda}\dot{r}^{\lambda}} + \Lambda\,(r_{\lambda}r^{\lambda})}{c}\right). \tag{3.53}$$

The tensor, consequently, is symmetric. On comparing its first component with the momentum-energy tensor of an ordinary (spinless) point particle we see that the constant M is the counterpart of the latter's mass. If we search to interprete it as the energy of the particle (in its own reference system), we find that it differs from the energy of the spinless particle by the term $\frac{\sqrt{-\dot{r}_{\lambda}\dot{r}^{\lambda}} + \Lambda r_{\lambda}r^{\lambda}}{c}$ which, accordingly, would appear to correspond to the energy characteristic of spin particles, the energy refered to by Mathisson as "accelerattional" [6].

The author wishes to thank Professor J. Weyssenhoff, Professor B. Średniawa and A. Staruszkiewicz, M. Sci., for their valuable discussions and numerous helpful hints.

REFERENCES

[1] Borelowski, Z., *Acta Phys. Polon.*, in press.
[2] Borelowski, Z., *Acta Phys. Polon.*, in press.
[3] Borelowski, Z., Średniawa, B. *Acta Phys. Polon.*, **24**, 609 (1964).
[4] Rayski, J., *Helv. Phys. Acta*, **36**, 1081 (1963).
[5] Fok, V., *Teoria Prostranstva Vremyeni i Tyagotyenya*, Moskva 1961.
[6] Mathisson, M., *Acta Phys. Polon.*, **6**, 163 (1937).