THERMODYNAMIC APPROXIMATION FOR TRANSVERSE MOMENTUM DISTRIBUTION OF SECONDARY PIONS

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The formulae for the transverse momentum distribution based on the thermodynamic approximation of phase-space integrals are applied to $\pi^{-}p$ interactions at 10 GeV. It is shown that the thermodynamic approximation with the correction of the order of n^{-1} agrees very well with the experimental data. The agreement is much better than for the transverse momentum distributions proposed up to now.

I. In this paper we analyse the shape of the transverse momentum distribution of secondaries in the $\pi^{-}p$ collisions at accelerator energies. This problem has been already discussed on the basis of some empirical formulae by several authors [2], [4], [7]. We propose to describe the transverse momentum distribution of secondaries by the formula following from the thermodynamical approximation to the statistical model (TF). The experimental data for the $\pi^{-}p$ collisions show a rather good agreement with TF if one considers the temperature as a free parameter, not related to the total energy of the system. Such concept of "transverse temperature" has been already proposed ([6], [8]) and we consider our result as supporting this hipothesis.

We have checked also that the TF represents a rather good approximation to the exact prediction of the statistical model for the transverse momentum distribution. Our conclusion is therefore that the transverse momentum distribution of secondaries in $\pi^{-}p$ collisions with moderate multiplicities (*e.g.* $n \geqslant 4$) can be calculated from the corresponding phase-space, if the energy of the phase-space is considered as a free parameter.

In Section II we discuss some of the transverse momentum distributions proposed up to now. In Section III we introduce the thermodynamic formula (TF) for transverse momentum distribution (*i.e.* the Boltzmann distribution integrated over the longitudinal momentum) and compare it with the transverse momentum distribution obtained from the statistical model. In Section IV we discusss the "transverse temperature". The dependence

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of the mean value of p_T on the multiplicity of produced particles is discussed in Section V. Our conclusions are given in the last Section.

II. Recently Bartke *et al.* [2] studied the distribution of transverse momentum for pions in eight and ten prong stars resulting from $\pi^- p$ collisions at 10 GeV. According to these authors, the empirical formulae

$$\varrho(p_T) = a^2 p_T e^{-a p_T} \tag{1}$$

and

$$\varrho(p_T) = 2a^2 p_T e^{-a^2 p_T^2} \tag{2}$$

disagree with the experimental distribution. (In these formulae a is an empirical parameter and the normalization is

$$\int_0^\infty \varrho(p_T) dp_T = 1). \tag{3}$$

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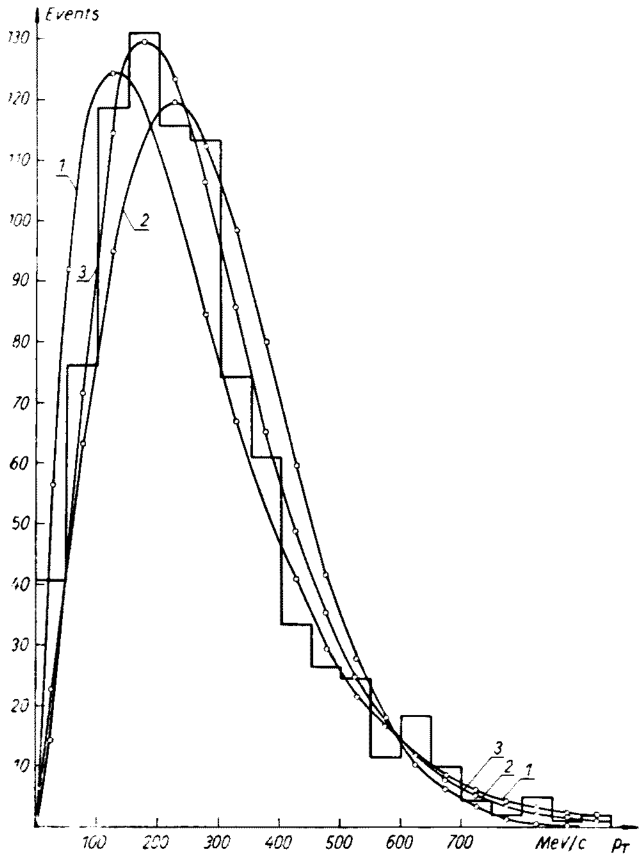

Fig. 1. The best fits for transverse momentum distributions (1) (the curve *1*), (2) (the curve *2*) and (4) (the curve *3*). The data are taken from the Ref. [2]

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We found that the formula

$$\varrho(p_T) = \frac{a^3}{2} p_T^2 e^{-ap_T} \tag{4}$$

is also inconsistent with the above data.

Distributions (1), (2), (4) are compared with the experimental histogram in Fig. 1. For the best fit curves χ^2 equals 39,8 ($a^{-1} = 125$ MeV/c, $\langle p_T \rangle = 250$ MeV/c); 79.2 ($a^{-1} =$ $= 312$ MeV/c, $\langle p_T \rangle = 274$ MeV/c) and 77 ($a^{-1} = 90.9$ MeV/c, $\langle p_T \rangle = 273$ MeV/c) which for 13 degrees of freedom corresponds to a rejection of all hypotheses at a confidence limit of below 10^{-3}. (The mean values for these distributions are $2a^{-1}$, $\frac{1}{2}\pi^{1/2} a^{-1}$ and $3a^{-1}$ respectively). The experimental value of $\langle p_T \rangle$ is 258^{+10}_{-6} MeV/c.

III. We noticed that the thermodynamic formula

$$\varrho(p_T) = N \int\limits_{-\infty}^{+\infty} 2\pi p_T e^{-\beta p_0} dp_L = \frac{\beta p_T \sqrt{p_T^2+m^2}\, K_1(\beta\sqrt{p_T^2+m^2})}{m^2 K_2(m\beta)}, \tag{5}$$

where

N — normalization constant,

$p_0 = (p_L^2+p_T^2+m^2)^{1/2}$,

m — pion mass,

$K_n(x)$ — MacDonald functions,

agrees very well with experiment, if β is interpreted as an empirical parameter. The best fit curve ($\beta^{-1} = 100$ MeV/c, $\langle p_T \rangle = 250$ MeV/c) is compared with the experimental histogram in Fig. 2. $\chi^2 = 15.7$ for 13 degrees of freedom, which means a good fit.

It is well known (*e. g.* Lurcat and Mazur [1], further quoted LM) that the thermodynamic approximation is a limiting case of the statistical model, valid at high multiplicities. Formula (5) corresponds to Fermi — noncovariant formulation of the model. In the thermodynamic approximation to the covariant version the integrand in (5) contains an additional factor p_0^{-1}.

Consequently

$$\varrho(p_T) = N' \int\limits_{-\infty}^{+\infty} \frac{2\pi p_T e^{-\beta p_0}}{p_0} dp_L = \frac{\beta p_T K_0(\beta\sqrt{p_T^2+m^2})}{m K_1(m\beta)}. \tag{6}$$

This distribution is compared with experiment in Fig. 2.

We obtain $\chi^2 = 21.33$ ($\beta^{-1} = 143$ MeV/c, $\langle p_T \rangle = 215$ MeV/c) which again means an acceptable fit (but for others distributions the noncovariant formula agrees better than the covariant one). Thus we find the TF formulae (5) or (6) certainly better than (1), (2) and (4).

In order to test the relation of formulae (5) and (6) to the exact predictions of the statis-

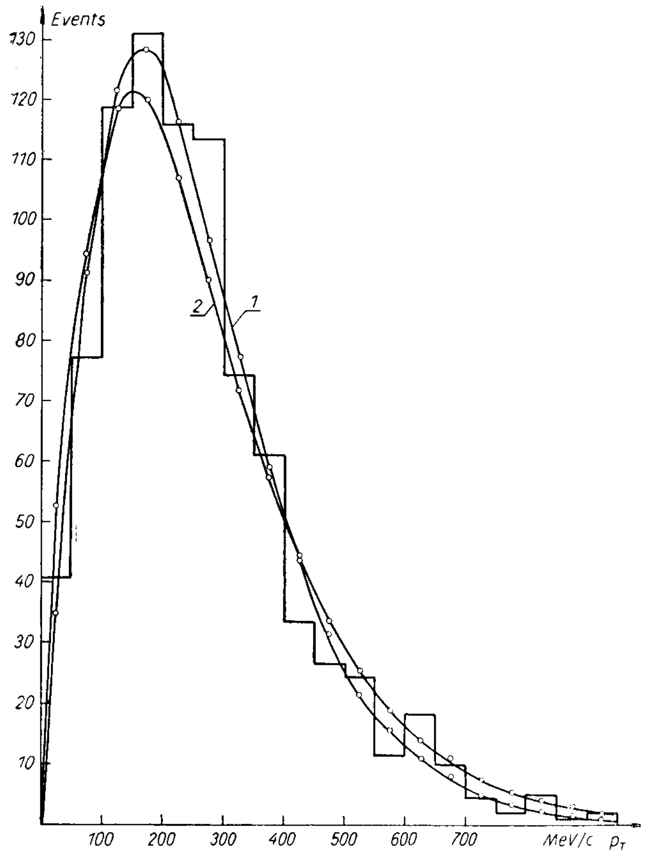

Fig. 2. The best fits for the transverse momentum distributions (5) (the curve *1*) and (6) (the curve *2*). The data are taken from the Ref. [2].

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tical model we evaluated higher order terms in the expansion of the statistical formula

$$\varrho(p_T) = \frac{\int\limits_{-\infty}^{+\infty} dp_L \int\limits_{0}^{\infty} d^{3(n-1)}p\,\delta^4(p - \sum p_i) \prod\limits_{i=1}^{n} u(p_{0i})\, 2\pi p_T}{\int d^{3n}p\,\delta^4(p - \sum p_i) \prod\limits_{i=1}^{n} u(p_{0i})}, \tag{7}$$

here

$u(p_0) = 1$ for the Fermi model,

$u(p_0) = p_0^{-1}$ for the covariant model.

Using the steepest descents method (see LM for details) one obtains

$$\varrho(p_T) = \varrho^{\mathrm{th}}(p_T)[1 + G_n(p_T) + O(n^{-2})], \tag{8}$$

where $\varrho^{\mathrm{th}}(p_T)$ is the thermodynamic approximation (5) or (6). $G_n(p_T)$ for each model is a combination of the MacDonald functions and elementary functions and is of the order of n^{-1}. Explicit formulae for $G_n(p_T)$ are given in the Appendix.

The corrected distributions are shown in Fig. 3 (for noncovariant model, for $\beta^{-1} =$

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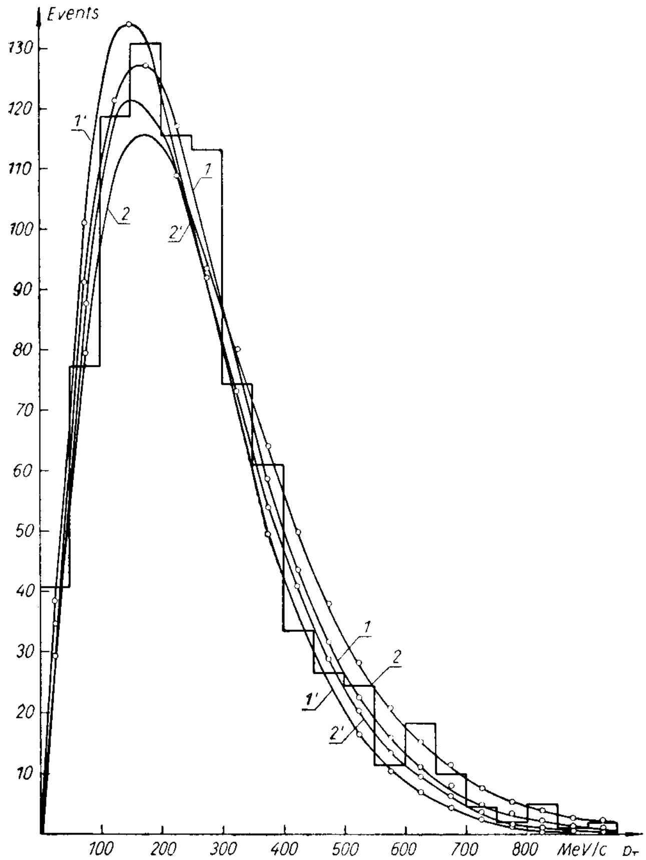

Fig. 3. The corrected transverse momentum distributions for $\beta^{-1} = 100$ MeV/c (the curve *1'* corrected to 1) and for β-1 111 MeV/c (the curve *2'* corrected to 2). The data are taken from the Ref. [2].

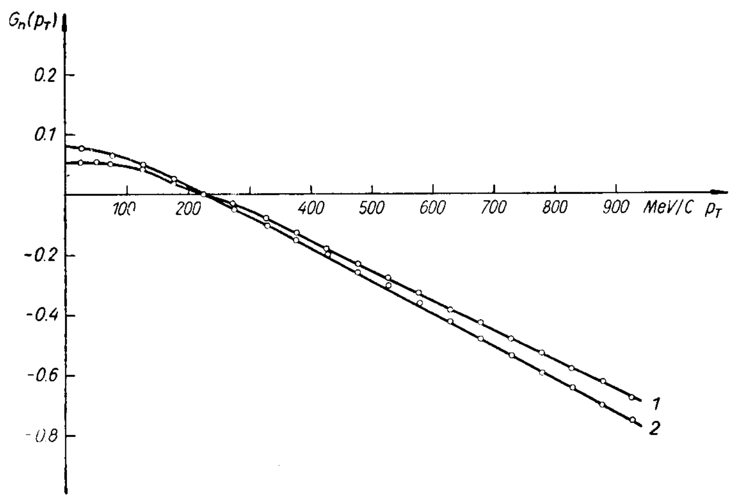

Fig. 4. The corrections for $G_n(p_T)$ for $\beta^{-1} = 100$ MeV/c for $N+8\pi$ production (the curve *1*) and 8π production (the curve *2*)

$= 100$ MeV/c and 111 MeV/c. The corrections were calculated for the $N+8\pi$ production at 10 GeV. From these plots one can see that the corrected formulae give a slightly better agreement with the experimental histogram. One can also see that the corrections are small. Consequently, the thermodynamic formulae may be considered as reasonable approximation to the prediction of the statistical model.

IV. We would like to make two additional remarks. 1. It is known (see *e. g.* LM) that for the total phase-space integral the thermodynamic approximation is very poor for reasonable multiplicities. Here it turns out to be good. The reason is that in the calculation of the total phase-space integral there is a large multiplicative correction term which corresponds to the transition from the thermodynamic approximation of the application of central limit theorem of theory of probabilities. This term does not appear when distributions for small subsystems (as for one particle) are evaluated. Consequently the thermodynamic model should yield good results for any one or even two particles distribution already at moderate multiplicities.

2. When applying the steepest descents method to the evaluation of phase-space integrals one obtains an equation which determines the temperature as a function of the total energy. In our analysis β was adjusted as an empirical parameter. The result is near to, but not identical with the result which would be obtained by solving the equation (A.11). For example for $N+8\pi$ production at 10 GeV $\beta^{-1} = 117$ MeV/c, whereas our $\beta^{-1} = 100$ MeV/c. This effect may be interpreted as a suggestion of the existance of a "transverse temperature" which may be used to find the transverse momentum distribution, but which is not identical with the equilibrium temperature. The idea is that a thermodynamic formula with a suitable temperature might be applicable to the transverse motion, while the longitudinal motion is very complicated and does not corresponds to the same equilibrium situation. References to the hydrodynamical model may be traced from Roinishvili [8], for a recent discussion see Hegedorn [5].

V. We have also fitted the transverse momentum distribution in $p+6\pi$ production in π^-p collision at 10 GeV [1]. Also in this case TF represents a better agreement with the experiment than all others ([1], [2] and [4]). The parameters β^{-1} for this reaction is given in Table I. It is seen that β^{-1} decreases with increasing multiplicity. The energy from Eq. (A.22) for each experimental parameter β is also given in Table I.

VI. Our conclusions can be formulated as follows.

We have investigated the transverse momentum distribution in π^-p inelastic collision in the region of few GeV/c of primary momentum. The experimental data have been compared with the approximate predictions of the statistical model.

The analysis shows that:

1. The thermodynamic formula for transverse momentum distribution agrees better with the experimental data than all others proposed up to now.

2. This formula represents a reasonable approximation to the transverse momentum distribution predicted by the statistical model. The first correction of the order of n^{-1} calculated by steepest descents method from the statistical model improves slightly the agreement with the experiment.

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TABLE I

Reaction	p_{Lab} GeV/c	E_0 GeV	β^{-1} MeV/c	$\langle p_T \rangle$ exp MeV/c	χ^2 for TF fit	$\langle \chi^2 \rangle$	confidence level	$\langle p_T \rangle$ th MeV/c	E GeV
$\pi^- + p \to 3\pi^- + 2\pi^- + p$	10	4.5	143	330 ± 12	10.4	18	0.92	312	3.6
$\pi^- + p \to 4\pi^- + 3\pi^+ + \pi^0 + p$	10	4.5	100	258 ± 10	15.7	13	0.28	250	4.0

E_0 — total CMS energy, $E_0 = \sqrt{2ME + m^2 + M^2}$, M — mass of the nucleon, E — energy from phase-space in CMS, $E — 3n\,\beta^{-1} + \sum_{i=1}^{n} m_i \frac{K_1(m_i\beta)}{K_2(m_i\beta)}$

3. The thermodynamical approximation of transverse momentum distribution from the noncovariant Fermi model gives a better agreement with experiment than from the covariant model.

4. The "transverse temperature" corresponding to the transverse momentum distribution differs from the equilibrium temperature.

5. The mean value of p_T (see also [3]) and the parameter β^{-1} both depend on the multiplicity in the investigated energy region (10 GeV).

The author would like to thank Dr K. Zalewski for suggesting this investigation and for many useful remarks as well as to Dr A. Białas for discussion. He also thanks Dr J. Bartke and J. Danysz for discussion of the experimental data.

APPENDIX

Using the steepest descents method (see LM) we obtain from (7)

$$\varrho(p_T) = \varrho^{\text{th}}(p_T)\,[1 + G_n(p_T) + O(n^{-2})], \tag{A.1}$$

where

$$G_n(p_T) = I_1 - I_2, \tag{A.2}$$

$$I_1 = \frac{1}{2}\left[\frac{e}{b} + \frac{3d}{a}\right], \tag{A.3}$$

$$I_2 = \frac{d}{2}\left[\frac{c}{b_2} + \frac{3(b-a)}{ab}\right] \tag{A.4}$$

and

$$a = -3n - \sum R(m_i\beta), \tag{A.5}$$

$$b = 3n - 3\sum R(m_i\beta) + \beta^2 \sum m_i^2, \tag{A.6}$$

$$c = -6n - 6\sum R(m_i\beta) + 3\beta^2 \sum m_i^2 - 9\sum [R(m_i\beta)]^2$$
$$+ 2\beta^2 \sum m_i^2 R(m_i\beta) - 2\sum [R(m_i\beta)]^3, \tag{A.7}$$

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$$d = 4 - \frac{\beta^2(p_T^2+m^2)}{R(\beta\sqrt{p_T^2+m^2})} + R(m\beta), \tag{A.8}$$

$$e = -4 + \frac{3\beta^2(p_T^2+m^2)}{R(\beta\sqrt{p_T^2+m^2})} + \beta^2 p_T^2 - \frac{\beta^4(p_T^2+m_i)^2}{[R(\beta\sqrt{p_T^2+m^2})]^2} +$$

$$+3R(m\beta)+[R(m\beta)]^2. \tag{A.9}$$

Here

$$R(x) = x\,\frac{K_1(x)}{K_2(x)}. \tag{A.10}$$

The condition for parameter β (see LM (2.28)) reads

$$3n + \sum R(m_i\beta) = \beta E. \tag{A.11}$$

One can see from these formulae that $G_n(p_T)$ is of order of n^{-1}.

For n particles with equal masses we have

$$G_n(p_T) = K(p_T)/n. \tag{A.12}$$

In practice we use the formulae (A.2)—(A.9).

In the similar way we calculate the correction for the momentum distribution given by covariant phase-space integral. The result is identical with (A.2)—(A.4) only the constants denote here

$$a = -2n - \sum R(m_i\beta)\cdot \tag{A.13}$$

$$b = 4n - \beta E + \beta^2 \sum m_i^2 - \sum [R(m_i\beta)]^2, \tag{A.14}$$

$$c = -4n + \beta^2 \sum m_i^2 + 2\beta^2 \sum m_i^2 R(m_i\beta) - \sum [R(m_i\beta)]^2$$

$$-2\sum [R(m_i\beta)]^3, \tag{A.15}$$

$$d = 2 - \frac{(p_T^2+m^2)\,\beta^2}{R(\beta\sqrt{p_T^2+m^2})} + R\,(m\beta), \tag{A.16}$$

$$e = -2 + \frac{(p_T^2+m^2)\,\beta^2}{R(\beta\sqrt{p_T^2+m^2})} + p_T^2\beta^2 - \frac{\beta^4(p_T^2+m^2)^2}{[R(\beta\sqrt{p_T^2+m^2})]^2} +$$

$$+R(m\beta)+[R(m\beta)]^2, \tag{A.17}$$

with

$$R(x) = x\,\frac{K_0(x)}{K_1(x)}. \tag{A.18}$$

The condition (A.11) reads now

$$2n + \sum R(m_i\beta) = \beta E \tag{A.19}$$

with the definition of R given by (A.18).

REFERENCES

[1] M. Bardadin, L. Michejda, S. Otwinowski, R. Sosnowski, *Institute of Nuclear Research Warsaw, report* 511(*V*) 1964; see also M. Bardadin-Otwinowska, L. Michejda, S. Otwinowski, R. Sosnowski, *Phys. Letters*, **21,** 351 (1966).

[2] J. Bartke, A. Eskreys, G. Pytkowicz, K. Zalewski, M. Bardadin, S. Otwinowski, R. Sosnowski, *Institute of Nuclear Research Warsaw, report* 597/VI/*PH*/1965.

[3] N. N. Biswas, I. Derado, N. Schmitz, W. Shephard, *Max-Planck Institute preprint.*

[4] G. Cocconi, L. J. Koester, D. H. Perkins, *Lawrence Radiation Laboratory, report UCRL* — 10022, 1961.

[5] R. Hagedorn, *Nuovo Cimento Suppl.*, **3,** 147 (1965).

[6] F. Lurcat, P. Mazur, *Nuovo Cimento*, **31,** 140 (1964).

[7] J. Orear, *Physics Letters*, **13,** 190 (1964).

[8] N. N. Roinishvili, *Zh. Eksper. Teor. Fiz.*, **41,** 919 (1961).

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