ON THE PARTICLE DECAYS IN THE THEORY WITH NON-COMPACT GROUPS

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The $SL(2, C)$ group is considered as the internal symmetry group. The vector-coupling coefficients for the irreducible unitary representation of $SL(2, C)$ are calculated. The matrix element for the decay of spin 1 particle into two spinless particles, is calculated.

1. *Introduction*

Recently a series of papers by Budini and Fronsdal [1], Fronsdal [2], [3], Rühl, Salam *et al.* [5], Nguyen van Hieu [6], [7], Dao Wong Duc and Nguyen van Hieu [8], [9] has been devoted to the application of non-compact groups to elementary particles. The non-compact symmetry group G is given as the semi-simple product of the Poincaré group P and some non-compact symmetry group:

$$G = P \rtimes S$$

In such a theory particles are classified according to some irreducible (infinite-dimensional) representation of the internal symmetry group S. For the group S, the groups $SL(2, C)$, $U(6,6)$ *etc.* can be chosen[1]. It is, in fact, possible to calculate the matrix elements of physical processes in the theory which is consistent with the unitarity condition. But the problem of calculation technique is non trivial one, because the considered symmetry groups are non-compact.

The attempts of performing such calculations were made by some authors. In particular, in the paper of Bisiacchi and Fronsdal [10] vector-coupling coefficients for the $SL(2, C)$ group were obtained under some restrictions imposed on the parameters determining the irreducible representations of this group. In the present paper a method of calculation of the vector coupling coefficients for the group $SL(2, C)$ is proposed which does not

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[1] In paper [7] the possibility of introducing the unitary S-matrix into the theory of infinite multiplets was considered.

impose any restrictions on the parameters determining irreducible unitary infinite-dimensional representation. Applying the matrix elements of the transition from the non-physical basis to the physical one of the little group of $SL\,(2,\,C)_p$, given in the paper of Dao Wong Duc and Nguyen van Hieu [9] we obtain the expressions for matrix elements of decay processes up to some scalar factor. In the last paragraph of the present paper we consider, as an example, the case of the decay of the spin 1 particle into two spinless particles. The result that the decay $\varrho\to 2\pi$ is not forbidden, is obtained by calculations of some higher orders in the power expansion in energy in contrast with the result obtained in [10].

The method of the present paper can be applied to the calculation of an arbitrary vertex containing particles corresponding to irreducible representations of $SL(2,\,C)$ both for principal series and for supplementary one. This method can be also applied to an arbitrary group $SL(n,\,C)$; we consider the group $SL(2,\,C)$ as the model to illustrate the method.

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2. *Vector-coupling coefficients for SL(2, C)*

We denote the set ν and ϱ of the parameters, characterizing irreducible unitary representations of $SL(2,\,C)$ by α. For the principal series ν is a non-negative integer or half-integer, ϱ is real; for supplementary series $\nu=0$, $-1\leqslant i\varrho\leqslant 0$, *i. e.* now ϱ is imaginary. We shall denote the basis vectors of the irreducible unitary representations of $SL(2,\,C)$ by $|\alpha, j, m\rangle$, where j characterizes the finite-dimensional irreducible reprezentation of the maximal compact subgroup of $SL(2,\,C)$. This basis will be called canonical basis.

In order to obtain the vector-coupling coefficients it is necessary to construct the invariant of the group. We start with constructing the formal expression for the invariant in the case of three state-vectors in the $SU(2)$ group:

$$I=\sum_{\substack{m_1,\,m_2,\,m_3\\ m_1+m_2+m_3=0}}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}|\alpha_1 j_1 m_1\rangle|\alpha_1 j_2 m_2\rangle|\alpha_3 j_3 m_3\rangle \tag{1}$$

where $\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}$ are 3-j symbols of Wigner, the invariant of $SL(2,\,C)$ is obtained by multiplying (1) by the vector-coupling coefficient $\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ j_1 & j_2 & j_3\end{bmatrix}$ and summing up over j_1, j_2, j_3:

$$J=\sum_{j_1 j_2 j_3}\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ j_1 & j_2 & j_3\end{bmatrix}\sum_{m_1\,m_2\,m_3}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}|\alpha_1 j_1 m_1\rangle|\alpha_2 j_2 m_2\rangle|\alpha_3 j_3 m_3\rangle \tag{2}$$

From the definition of an invariant of the group we have for each generator X_i of this group the relation

$$X_i J=0$$

This relation is, of course, fulfilled for the compact generators H_i of $SL(2,\,C)$. Since non-compact generators can be expressed by the commutators $[H_\pm, F_3]$, it is sufficient to utilize only the relation

$$F_3\,J=0 \tag{3}$$

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where F_3 operates on the basis vectors as follows (see ref. [11])

$$F_3|\alpha jm\rangle = C^{\alpha}_{m}(j, j-1)|\alpha, j-1, m\rangle + \\ + C^{\alpha}_{m}(j, j)|\alpha jm\rangle + C^{\alpha}_{m}(j, j+1)|\alpha, j+1, m\rangle \tag{4}$$

F_3 applied to (2) denotes the function

$$F_3 = F_3^{(1)} + F_2^{(2)} + F_3^{(3)} \tag{4'}$$

where $F_3^{(i)}$ operates in the space of vectors $|\alpha_i j_i m_i\rangle$. Inserting (2) into (3) and using (4) and (4') we obtain the recursion relation for vector-coupling coefficients

$$\begin{aligned}
&C^{\alpha_1}_{m_1}(j_1+1, j_1) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1+1 & j_2 & j_3 \end{bmatrix} \begin{pmatrix} j_1+1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_1}_{m_1}(j_1, j_1) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2 & j_3 \end{bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_1}_{m_1}(j_1-1, j_1) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1-1 & j_2 & j_3 \end{bmatrix} \begin{pmatrix} j_1-1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_1}_{m_1}(j_2+1, j_2) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2+1 & j_3 \end{bmatrix} \begin{pmatrix} j_1 & j_2+1 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_2}_{m_2}(j_2, j_2) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2 & j_3 \end{bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_2}_{m_2}(j_2-1, j_2) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2-1 & j_3 \end{bmatrix} \begin{pmatrix} j_1 & j_2-1 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_3}_{m_3}(j_3+1, j_3) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2 & j_3+1 \end{bmatrix} \begin{pmatrix} j_1 & j_2 & j_3+1 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_3}_{m_3}(j_3-1, j_3) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2 & j_3 \end{bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + \\
&+ C^{\alpha_3}_{m_3}(j_3-1, j_3) \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ j_1 & j_2 & j_3-1 \end{bmatrix} \begin{pmatrix} j_1 & j_2 & j_3-1 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0.
\end{aligned} \tag{5}$$

Here

$$\begin{gathered}
C^{\alpha}_{m}(j-1, j) = \sqrt{j^2-m^2}\; C_j, \\
C^{\alpha}_{m}(j+1, j) = -\sqrt{(j+1)^2-m^2}\; C_{j+1} \\
C^{\alpha}_{m}(j, j) = -m\, A_i, \\
A_j = \frac{\nu\varrho}{j(j+1)}, \quad C_j = \frac{i}{j}\sqrt{\frac{(j^2-\nu^2)(j^2+\varrho^2)}{4j^2-1}}
\end{gathered} \tag{6}$$

From the recursion formula (5) we can, in principle, obtain explicit expression for arbitrary vector-coupling coefficient $\begin{bmatrix} \alpha_1\, \alpha_2\, \alpha_3 \\ j_1\; j_2\; j_3 \end{bmatrix}$. Since the calculations are cumbersome, we give here only expressions for vector-coupling coefficients for the low values of j_i. We remark, that the coefficients $\begin{bmatrix} \alpha_1\, \alpha_2\, \alpha_3 \\ 0\;\; 0\;\; 0 \end{bmatrix}$ and $\begin{bmatrix} \alpha_1\, \alpha_2\, \alpha_3 \\ \frac{1}{2}\;\; \frac{1}{2}\;\; 0 \end{bmatrix}$ remain arbitrary; we normalize them to unity;

$$\Delta_i(j) = \sqrt{(j-\nu_i^2)(j^2+\varrho^2)}, \qquad [j_1 j_2 j_3] \equiv \begin{bmatrix} \alpha_1\, \alpha_2\, \alpha_3 \\ j_1\; j_2\; j_3 \end{bmatrix}$$

$$[0\;1\;1] = \frac{\sqrt{3}}{2}\, \frac{[\Delta_1(1)]^2-[\Delta_2(1)]^2-[\Delta_3(1)]^2}{\Delta_2(1)\,\Delta_3(1)}$$

$$[1\;0\;1] = \frac{\sqrt{3}}{2}\, \frac{[\Delta_2(1)]^2-[\Delta_3(1)]^2-[\Delta_1(1)]^2}{\Delta_3(1)\,\Delta_1(1)}$$

$$[1\;1\;0] = \frac{\sqrt{3}}{2}\, \frac{[\Delta_3(1)]^2-[\Delta_1(1)]^2-[\Delta_1(1)]^2}{\Delta_1(1)\,\Delta_2(1)}$$

$$[1\;1\;1] = \frac{i}{2\sqrt{2}} \left\{ \frac{\varrho_2\nu_2-\varrho_1\nu_1}{\Delta_2(1)\,\Delta_3(1)}\, \Delta_1(1) + \frac{\varrho_3\nu_3-\varrho_1\nu_1}{\Delta_3(1)\,\Delta_1(1)}\, \Delta_2(1) + \frac{\varrho_1\nu_1-\varrho_2\nu_3}{\Delta_1(1)\,\Delta_2(1)}\, \Delta_3(1) \right\}$$

$$[2\;1\;1] = \left\{ -\frac{i\sqrt{3}}{2} (\nu_2\varrho_2-\nu_3\varrho_3)[1\;1\;1] + \frac{3}{2\sqrt{2}} \left[\Delta_2(1)[1\;0\;1] + \right.\right.$$
$$\left.\left. +\Delta_3(1)[1\;1\;0] - \frac{1}{3}\, \Delta_1(1)[0\;1\;1] \right]\right\} \frac{1}{\Delta_1(2)}$$

$$[1\;2\;1] = \left\{ -\frac{i\sqrt{3}}{2} (\nu_3\varrho_3-\nu_1\varrho_1)[1\;1\;1] + \frac{3}{2\sqrt{2}} \left[\Delta_3(1)[1\;1\;0] + \right.\right.$$
$$\left.\left. +\Delta_1(1)[0\;1\;1] - \frac{1}{3}\, \Delta_2(1)[1\;0\;1] \right]\right\} \frac{1}{\Delta_2(2)}$$

$$[1\;1\;2] = \left\{ -\frac{i\sqrt{3}}{2} (\nu_1\varrho_1-\nu_2\varrho_2)[1\;1\;1\,] + \frac{3}{2\sqrt{2}} \left[\Delta_1(1)[0\;1\;1] + \right.\right.$$
$$\left.\left. +\Delta_2(1)[1\;0\;1] - \frac{1}{3}\, \Delta_3(1)\,[1\;1\;0] \right]\right\} \frac{1}{\Delta_3(2)}$$

$$\begin{bmatrix} \frac{1}{2}\; \frac{1}{2}\; 1 \end{bmatrix} = \frac{i}{3\sqrt{2}}\, \frac{\nu_2\varrho_2-\nu_1\varrho_1}{\Delta_3(1)} \begin{bmatrix} \frac{1}{2}\; \frac{1}{2}\; 0 \end{bmatrix}$$

The symbols $\begin{bmatrix} \frac{1}{2}\; 1\; \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 1\; \frac{1}{2}\; \frac{1}{2} \end{bmatrix}$ are obtained by cyclic permutation.

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3. *Matrix elements for decays*

Now we apply the results of the previous paragraph to the calculation of the decays of particles belonging to infinite multiplets of $SL(2, C)$. We have shown, that the basic vectors of the irreducible unitary representations of the group S can be expressed in the form $|\alpha jm\rangle$.

Since the symmetry group G is the semi-simple product of the Poincaré group P and the internal symmetry group S, the state vector can be represented in the form

$$|ps, \alpha jm\rangle = |ps\rangle \otimes |\alpha jm\rangle.$$

The basis formed by these vectors, called in [7] the canonical basis is the non-physical one. We pass to the rest system by the Lorentz transformation $\lambda_{p\leftarrow\hat{p}}$:

$$|ps, \alpha jm\rangle = U(\lambda_{p\leftarrow\hat{p}})|\hat{p}, \alpha jm\rangle = U^{p}(\lambda_{p\leftarrow\hat{p}})|\hat{p}s\rangle \otimes U^{s}(\lambda_{p\leftarrow\hat{p}})|\alpha jm\rangle.$$

The vectors on the r. h. s. of this equation will be denoted by $|ps, \alpha\tilde{j}\tilde{m}\rangle$. We shall consider the case $s = 0$ only. Then the physical basis can be expressed by the vector $|p, \alpha jm\rangle$ in the following way

$$|p, \alpha\tilde{j}\tilde{m}\rangle = D^{\alpha}_{jm,j'm'}(\lambda_{p\leftarrow\hat{p}})|p, \alpha\, j'm'\rangle.$$

Since the state vectors $|p, \alpha\tilde{j}\tilde{m}\rangle$ have the form of the direct product given above, we have finally

$$|\alpha\tilde{j}\tilde{m}\rangle = D^{\alpha}_{jm,j'm'}(\lambda_{p\leftarrow\hat{p}})|\alpha j'm'\rangle. \tag{7}$$

The coefficients $D^{\alpha}_{jm,j'm'}$ have been obtained in the paper by Dao Wong Duc ans Nguyen van Hieu [9] and have the form

$$\begin{aligned} D^{\nu\varrho}_{jm,j'm'} = {} & \frac{\delta_{mm'}}{j+j'+1}\{(2j+1)(2j'+1)(j+m)!(j-m)!\cdot \\ & \cdot(j+\nu)!(j-\nu)!(j'+m)!(j'-m)!(j'+\nu)!(j'-\nu)!\}^{1/2}\cdot \\ & \cdot\sum_{d,d'}(-1)^{d+d'}\frac{(d+d'+m+\nu)!(j+j'-d-d'-m-\nu)!}{d!d'!(j-m-d)!(j'-m-d')!(\nu+m+d)!(\nu+m'+d')!(j-\nu-d)!(j'-\nu-d')!} \\ & \cdot\varepsilon^{2\left(2d'+m+\nu+1+\frac{i\varrho}{2}\right)}F\left(j'+1+\frac{i\varrho}{2}, d+d'+m+\nu+1; j+j'+2, 1-\varepsilon^{4}\right), \end{aligned} \tag{8}$$

where F is the hypergeometric function. Using (8) we write out the matrix elements of the decay of a particle into two particles belonging to the infinite multiplet of the group $SL(2, C)$ as follows

$$\langle q_1\nu_1\varrho_1 j_1 m_1; q_2\nu_2\varrho_2 j_2 m_2|T|\hat{p}\;\nu_3\varrho_3 j_3 m_3\rangle$$

$$= \sum_{j'j''}\sum_{i_1m_1,j'm'} D^{\nu_1\varrho_1}_{j_1m_1,j'm'}(q_1)\,D^{\nu_2\varrho_2}_{j_2m_2,j'n''}(q_2)\cdot\begin{bmatrix}(\nu_1\varrho_1) & (\nu_2\varrho_2) & (\nu_3\varrho_3)\\ j' & j'' & j_3\end{bmatrix}\begin{pmatrix} j' & j'' & j_3\\ m' & m'' & m_3\end{pmatrix}F(s, t) \tag{9}$$

We consider now, as an example, the decay of a spin 1 particle into two spin 0 particles. We restrict ourselfs to the case, when $\nu_1 = \nu_2 = 0$, $\varrho_1 = \varrho_2$ and $\nu_3 = 0.1$. Let us denote by I_0 the contribution of the case $\nu_3 = 0$ and by I the contribution of $\nu_3 = 1$ to the matrix element. We get

$$I_0 = D^{0\varrho_1}_{00,00} D^{0\varrho_1}_{00,00} \begin{bmatrix} \varrho_1 \, \varrho_2 \, \varrho_3 \\ 0 \; 0 \; 0 \end{bmatrix} \begin{pmatrix} 0\,0\,1 \\ 0\,0\,0 \end{pmatrix} + D^{0\varrho_1}_{00,10} D^{0\varrho_1}_{00,00} \begin{bmatrix} \varrho_1 \, \varrho_2 \, \varrho_3 \\ 1 \; 0 \; 1 \end{bmatrix} \begin{pmatrix} 1\,0\,1 \\ 0\,0\,0 \end{pmatrix} +$$

$$+ D^{0\varrho_1}_{00,20} D^{0\varrho_1}_{00,00} \begin{bmatrix} \varrho_1 \, \varrho_2 \, \varrho_3 \\ 2 \; 0 \; 1 \end{bmatrix} \begin{pmatrix} 2\,0\,1 \\ 0\,0\,0 \end{pmatrix} + \ldots$$

$$+ D^{0\varrho_1}_{00,00} D^{0\varrho_1}_{0,10} \begin{bmatrix} \varrho_1 \, \varrho_2 \, \varrho_3 \\ 0 \; 1 \; 1 \end{bmatrix} \begin{pmatrix} 0\,1\,1 \\ 0\,0\,0 \end{pmatrix} + D^{0\varrho_1}_{00,00} D^{0\varrho_1}_{00,00} \begin{bmatrix} \varrho_1 \, \varrho_2 \, \varrho_3 \\ 0 \; 2 \; 1 \end{bmatrix} \begin{pmatrix} 0\,2\,1 \\ 0\,0\,0 \end{pmatrix} + \ldots \tag{10}$$

+ terms of higher order in j, which will not be taken into account.

Performing the calculations in (10) we get for I_0

$$I_0 = \left(\frac{1+\varrho_3^2}{1+\varrho_1^2} \right)^{1/2} \cdot$$

$$\left[\varepsilon^{4\left(1-\frac{i\varrho_1}{2}\right)} F\left(2+\frac{i\varrho_1}{2}\,; 1; 3; 1-\varepsilon^4\right) F\left(1+\frac{i\varrho_1}{2}\,, 1; 2; 1-\varepsilon^4\right) + \right.$$

$$\left. + \varepsilon^{2\left(3-\frac{i\varrho_1}{2}\right)} F\left(2+\frac{i\varrho_1}{2}\,; 2; 3; 1-\varepsilon^4\right) F\left(1+\frac{i\varrho_1}{2}\,, 1; 2; 1-\varepsilon^4\right) + \ldots \right.$$

I_1 can be obtained from I_0 by changing $v_3 = 0$ by $\nu_3 = 1$. We obtain at once

$$I_1 = 0$$

4. *Results and discussion*

The recursion relation (5) derived in section *2* allows to obtain exact values of vector--coupling coefficients for $SL(2, C)$ with the help of simple (but somewhat cumbersome) algebraic operations.

The functions D (see (8)) used in section *3* in the construction of matrix elements contain hypergeometrical functions. But the series in (9) is slowly convergent for arbitrary energies ε. Therefore in order to apply this theory to the investigation of physical processes it is necessary to consider only low-energy processes. But the same conditions are necessary for preserving the exact symmetry. Therefore we restricted ourselfs to the first two terms in the calculations of (11). Let us remark that if the group $SU(2)$ as the group of internal symmetry is used, the $\varrho \to 2\pi$ decay is forbidden. But if we apply the little group $SU(2)_p$ as internal symmetry group, the decay $\varrho \to 2\pi$ is allowed. This means that the group $SU(2)_p$ takes into account the orbital angular momentum.

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