TIME EVOLUTION OF A SYSTEM WITH A STOCHASTIC HAMILTONIAN AND A GIVEN INITIAL STATE

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The time evolution of a quantum system, which with a given Hamiltonian and an initial state is described by the Von Neumann equation, is discussed in the case of a stochastic Hamiltonian. The notion of a stochastic Hamiltonian is taken from the statistical theory of the energy levels of complex systems (*cf.* Porter 1965). Following the ideas of Ingarden (1965, independently also Porter 1965, Bronk 1966), the conventional statistical approach used in the statistical theory of energy levels is generalized by means of the variational principle of information thermodynamics (Ingarden 1963). The optimum density operator is introduced and its matrix elements are calculated in some special cases. The reversibility of the evolution of the system described by the optimum density operator is discussed. It is shown that in the cases considered this evolution is irreversible.

1. *Introduction*

The idea of a stochastic Hamiltonian appeared first in nuclear physics, where, in general, we don't know the exact form of Hamiltonian. We only know some symmetry properties of the hypothetical Hamiltonian and some statistical properties of nuclei spectra, such as the distribution of neutron-widths, energy levels spacings, fission widths, *etc.* In the absence of any precise knowledge of the Hamiltonian, one assumes a reasonable probability distribution for its matrix elements (stochastic matrix hypothesis), from which one deduces statistical properties of its spectrum. Such theory, known as the statistical theory of the energy levels of complex systems, was developed by Landau 1955; Wigner 1957, 1958; Rosenzweig 1960, 1962; Porter 1960; Mehta 1960; Dyson 1962 a–c and many others (see also Porter 1965).

Another type of stochastic Hamiltonian is utilized when in macroscopic systems some exterior parameters change stochastically. The Hamiltonian has a time dependent, stochastic part, *e.g.*,

$$H(t) = H_0(t)+s(t)V, \tag{1.1}$$

where $s(t)$ is a stochastic function (*cf.*, *e.g.*, Primas 1961).

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In the present paper the first type of stochastic Hamiltonians is used, *i.e.* the Hamiltonian is considered as a Hermitian matrix of a very large order N with its matrix elements distributed at random. The Hamiltonian is assumed time independent.

2. *The optimum density operator*

In quantum statistics the state of a system is characterized by a density operator ϱ. The time evolution of a density operator is described by the Von Neumann equation

$$i\frac{\partial\varrho(t)}{\partial t} = [H, \varrho(t)]. \tag{2.1}$$

For a time independent Hamiltonian the solution of (2.1) is given by

$$\varrho(t, H) = e^{-iHt}\varrho(0)e^{iHt}. \tag{2.2}$$

We treat the density operator ϱ (0) as fixed and the Hamiltonian as random. The statistical approach with regard to the density operator was also investigated (*cf.* Schwegler 1965). If we assume the stochastic matrix hypothesis, the Hamiltonian H must belong to the set of operators admissible (allowable) for the given system.

Let us consider the set $\mathcal{H}'_N$ of all admissible Hamiltonians of the system which is a subset of the set $\mathcal{H}_N$of all N-dimensional Hermitian operators. For the sake of simplicity we assume, however, that the Hilbert space of the system is finite-dimensional, as is always done in the statistical theories of energy levels. The set $\mathcal{H}'_N$ is fixed by the physical conditions of the problem, *e.g.*, in nuclear physics one usually considers the set od real symmetric operators because of the required time reversal invariance. We consider the two types of $\mathcal{H}'_N$ only: I) the set $\mathcal{H}^r_N$ of N-dimensional, real symmetric operators, II) the set $\mathcal{H}^c_N$ of N-dimensional complex Hermitian operators. (Besides, several authors considered the set $\mathcal{H}^q_N$ of quaternion operators, *cf.* Dyson 1962a, also Ginibre 1965.)

Let us denote the volume element of $\mathcal{H}'_N$ by d^nH. *E.g.* the volume element of $\mathcal{H}^r_N$ is

$$d^nH = \prod_{i\leqslant j} dH_{ij}, \; ||H_{ij}|| = H \in \mathcal{H}^r_N, \; (i, j = 1, ..., N). \tag{2.3}$$

Let η be the σ-algebra of all subsets of $\mathcal{H}'_N$ and $P(E)$ be a probability measure on η,

$$P(E) = \int_E P(d^nH) = \int_E p(H)d^nH, \; (E \in \eta), \tag{2.4(}$$

i.e. we have a probability space $(\mathcal{H}'_N, \eta, P)$, (*cf.*, *e.g.*, Rosenblatt 1962, Chap. IV). If we know the probability measure $P(E)$, we can describe all the stochastic properties of the system, connected with a random Hamiltonian. In particular, we can deal — either with the distribution of energy eigenvalues which has been the subject of the statistical theory of energy levels (stationary problem), — or with the time evolution of the system, which is the subject of the present investigation (evolutionary problem).

To the set $\mathcal{H}'_N$ there corresponds a set $\mathcal{R}_N$ of solutions (2.1) of the form (2.2), where

$$VH \in \mathcal{H}'_N, \mathcal{I}\varrho(t, H) \in R_N. \tag{2.5}$$

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The set $\mathcal{R}_N$ forms the "set of all possible evolutions" for a given $\varrho(0)$. For a stochastic theory we introduce the mean density operator over $\mathcal{H}'_N$ as

$$\langle\varrho(t)\rangle_p = \int_{\mathcal{H}_N} \varrho(t, H)\, P(d^nH) = \int_{\mathcal{H}'_N} \varrho(t, H) p(H) d^nH. \tag{2.6}$$

We know that the usual mean value of an arbitrary operator A is given by

$$\bar{A}(t, H) = \mathrm{Tr}\,\{\varrho(t, H)A\}. \tag{2.7}$$

Having (2.6) we may introduce the supermean value for A

$$\langle\bar{A}(t)\rangle_p = \mathrm{Tr}\,\{\langle\varrho(t)\rangle_p A\} = \int_{\mathcal{H}'_N} \mathrm{Tr}\,\{\varrho(t, H)\, A\} P(d^nH). \tag{2.8}$$

As $p(H)$, there can be chosen the most likely (optimum) probability distribution p_0 corresponding to the given knowledge about the set $\mathcal{H}'_N$, *i.e.* such that the entropy (information)

$$S[p] = -\int_{\mathcal{H}'_N} p(H) \log p(H)\, d^nH \tag{2.9}$$

attains its maximum value, *cf.* Ingarden 1965, p. 794. We now introduce the optimum density operator over $\mathcal{H}'_N$, substituting in (2.6) the optimum distribution p_0

$$\langle\varrho(t)\rangle_{p_0} = \int_{\mathcal{H}'_N} \varrho(t, H) p_0(H) d^nH. \tag{2.10}$$

We assume here that the optimum density operator describes the most probable behaviour of a system (the most probable state) with a stochastic Hamiltonian at the time t. Of course,

$$\langle\varrho(0)\rangle_{p_0} = \varrho(0),\ \mathrm{Tr}\,\langle\varrho(t)\rangle_{p_0} = \mathrm{Tr}\,\varrho(0) = 1. \tag{2.11}$$

When the probability distribution is of the form

$$p(H) = \delta(H-H_0), \tag{2.12}$$

we obtain from (2.6)

$$\langle\varrho(t)\rangle_p = \varrho(t, H)_0, \tag{2.13}$$

i.e. the usual expression (2.2).

3. Calculations with some optimum probability distributions

Every operator $H \in \mathcal{H}'_N$ can be diagonalized,

$$H = XDX^+,\ XX^+ = X^+X = I,\ D = ||\delta_{jk}\lambda_j||, \tag{3.1}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of H. Owing to (2.2), (2.10) and (3.1) the optimum density operator can be written in the form

$$\langle\varrho(t)\rangle_{p_0} = \langle Xe^{-iDt}X^+\varrho(0)Xe^{iDt}X^+\rangle_{p_0}, \tag{3.2}$$

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where

$$(e^{iDt})_{pq} = \delta_{pq} e^{i\lambda_q t} (p, q = 1, \ldots, N). \tag{3.3}$$

The matrix elements of (3,2) are

$$\langle \varrho(t)_{jk} \rangle_{p_0} = \langle \sum_{pqrs=1}^{N} X_{jq} X^*_{rq} X_{sp} X^*_{kp} e^{it(\lambda_p - \lambda_q)} \varrho_{rs}(0) \rangle_{p_0}. \tag{3.4}$$

The operators X form a unitary group U_N, if $H \in \mathcal{H}^c_N$, and an orthogonal group O_N, if $H \in \mathcal{H}^r_N$.

Let us calculate the matix elements of the optimum density operator in some special cases. We shall condsier the following situations:

a) the operator X is random (stochastic), but the eigenvalues λ_i are fixed, ($j = 1, \ldots, N$),

b) the eigenvalues λ_j are random, but the operator X is fixed,

c) both X and λ_j's are random.

Case a. In this case the probability measure $P(u)$ (or $P(0)$) is defined on the unitary group U_N (or on O_N), *e.g.*,

$$P(o) = \int_0 p(R)\mu(dR), \; (R \in o \in O_N), \tag{3.5}$$

where $\mu(dR)$ is the invariant measure of the orthogonal group, (*cf.* Weyl 1940, § 17, *cf.* also the "invariant integration", Gelfand 1958, § 1. 3, or Pontryagin 1954, § 29). We define the entropy with respect to the measure μ (μ-entropy) by

$$S[p] = -\int_{O_N} p(R) \log p(R) \mu\,(dR). \tag{3.6}$$

The entropy (3.6) attains its maximum for the homogeneous distribution over O_N

$$p_0(R) = 1, \; (\int_{O_N} \mu(dR) = 1). \tag{3.7}$$

The matrix elements of (3.4) are ($X \to R$)

$$\langle \varrho_{jk}(t) \rangle_{p_0(R)} = \sum_{pqrs=1}^{N} \langle R_{jp} R_{qp} R_{rs} R_{ks} \rangle\, O_N e^{i(\lambda_s - \lambda_p)t} \varrho_{qr}(0), \tag{3.8}$$

where

$$\langle R_{jp} R_{qp} R_{rs} R_{ks} \rangle_{O_N} = \int_{O_N} R_{jp} R_{qp} R_{rs} R_{ks} \mu(dR). \tag{3.9}$$

Substituting into (3.8) the results of the averaging (3.9), (see App. I, *cf.* also Ullah and Porter 1963), we obtain

$$\langle \varrho_{jk}(t) \rangle_{p_0} = \frac{1}{N+2} [\varrho_{jk}(0) + \varrho_{kj}(0)] +$$

$$+ \frac{2}{N(N-1)\,(N+2)} [(N-1)\varrho_{jk}(0) - \varrho_{kj}(0)] \Phi(t), \tag{3.10}$$

when $j \neq k$ and

$$\langle \varrho_{jj}(t)\rangle_{p_0} = \frac{1}{N+2}\left\{[2\varrho_{jj}(0)+\delta_{jj}] + \frac{2}{N-1}\left[\varrho_{jj}(0) - \frac{\delta_{jj}}{N}\right]\Phi(t)\right\}, \tag{3.11}$$

when $j = k$.

Here,

$$\Phi(t) = \sum_{1\leqslant p<q\leqslant N} \cos(\lambda_p-\lambda_q)t. \tag{3.12}$$

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A similar reasoning for the unitary group ($X \to U$) gives

$$\langle \varrho(t)_{jk}\rangle_{p_0}(U) = \sum_{pqrs=1}^{N} \langle U_{jq}U^*_{rq}U_{sp}U^*_{kp}\rangle_{U_N} e^{it(\lambda_p-\lambda_q)}\varrho_{ts}(0). \tag{3.13}$$

The averages $\langle U_{jq}U^*_{rq}U_{sp}U^*_{kp}\rangle_{U_N}$ are calculated in App. II. After substituting these averages in (3.13) one obtains

$$\langle \varrho_{jj}(t)\rangle_{p_0(U)} = \frac{1}{N+1}\left\{[\varrho_{jj}(0)+\delta_{jj}] + \frac{2}{N(N-1)}[N\varrho_{jj}(0)-\delta_{jj}]\Phi(t)\right\}, \tag{3.14}$$

for $j = k$ and

$$\langle \varrho_{jk}(t)\rangle_{p_0(U)} = \frac{1}{N+1}\varrho_{jk}(0)\left[1 + \frac{2}{N-1}\Phi(t)\right]. \tag{3.15}$$

for $j \neq k$. If the opertor $\varrho(0)$ is diagonal, then the formulae (3.10), (3.11) can be rewritten in the form

$$\langle \varrho(t)\rangle_{p_0} = A(t)\varrho(0) + a(t)I, \tag{3.16}$$

where

$$A(t) = \frac{2}{N+2}\left(1 + \frac{\Phi(t)}{N-1}\right), \qquad a(t) = \frac{1-A(t)}{N}. \tag{3.17}$$

The formulae (3.14, 3.15) can also be written in the form (3.16), with

$$A(t) = \frac{1}{N+1}\left[1 + \frac{2}{N-1}\Phi(t)\right]. \tag{3.18}$$

In this case we do not assume that $\varrho(0)$ is diagonal. Transformations like (3.16) are known as the "dynamical mapping" (Sudarshan, *et al.*, 1961, Jordan 1961, then Bausch 1965, Schlögl 1965).

Case b. We have the operator $R \in O_N$ (or $U \in N_N$) fixed and now the eigenvalues of the Hamiltonian are random variables. We can choose the basis R (or U) such, that all the Hamiltonians will be of the form

$$D = ||\lambda_j\delta_{jk}||, \ (j, k = 1, ..., N) \tag{3.19}$$

and the density operator $\varrho(t, D)$ becomes

$$\varrho(t, D) = e^{-iDt}\varrho(0)e^{iDt} \tag{3.20}$$

with its matrix elements in the form

$$\varrho_{jk}(t, \lambda_1, ..., \lambda_N) = \varrho_{jk}(0)e^{i(\lambda_k-\lambda_j)t}. \tag{2.21}$$

The entropy (2.9) becomes now

$$S[p] = -\int_{\mathcal{D}} p(\lambda_1, ..., \lambda_N) \log p(\lambda_1, ..., \lambda_N)d\lambda_1, ..., d\lambda_N, \tag{3.22}$$

$$\mathcal{D} = \{\langle\lambda_1, ..., \lambda_N\rangle\}.$$

If we assume that the eigenvalues λ_j' s are "spread out", *i.e.* if

$$\lambda_j \in [\Lambda_j - \Delta_j, \Lambda_j + \Delta_j], \quad (j = 1, ..., N) \tag{3.23}$$

where Λ_j's and Δ_j's are given numbers, the optimum distribution is of the form

$$p_0(\lambda_1, ..., \lambda_N) = \prod_{j=1}^{1} \frac{1}{2\Delta_j}, \tag{3.24}$$

Now we calculate the matrix elements of the optimum density operator

$$\langle\varrho_{jk}(t)_{p_0} = \int_{\Lambda_1-\Delta_1}^{\Lambda_1+\Delta_1} .. \int_{\Lambda_N-\Delta_N}^{\Lambda_N+\Delta_N} \varrho_{jk}(0)e^{i(\lambda_k-\lambda_j)t} p_0 d\lambda_1 ... \lambda_N. \tag{3.25}$$

After performing the integration we have

$$\langle\varrho_{jk}(t)\rangle_{p_0} = \begin{cases} \varrho_{jj}(0) & \text{for } j = k, \\ \dfrac{\sin\Delta_k t}{\Delta_k} \dfrac{\sin\Delta_j t}{\Delta_j} \dfrac{1}{t^2} e^{i(\Lambda_k-\Lambda_j)t}\varrho_{jk}(0), & j \neq k. \end{cases} \tag{3.26}$$

Alternatively, if we assume that the statistical moments,

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \lambda_j p(\lambda_1, ..., \lambda_N)d\lambda_1 ... d\lambda_N = \Lambda_j, \tag{3.27}$$

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \lambda^2 p(\lambda_1, ..., \lambda_N)d\lambda_1 ... d\lambda_N = \sigma_j^2 + \Lambda_j^2, \quad (j = 1, ..., N) \tag{3.28}$$

of λ_j' s are known, the optimum distribution is of the form

$$p_0 = (2\pi)^{-\frac{N}{2}} \prod_{j=1}^{N} \frac{1}{\sigma_j} \exp\left[-\frac{1}{2\sigma_j^2}(\lambda_j - \Lambda_j)^2\right]. \tag{3.29}$$

Averaging (3.21) with the distribution (3.29) one obtains for $j = k$,

$$\langle\varrho_{jk}(t)\rangle_{p_0} = \begin{cases} \varrho_{jj}(0) & \text{for } j = k, \\ e^{i(\Lambda_k-\Lambda_j)t} e^{-\frac{1}{2}(\sigma_j^2+\sigma_k^2)t^2}\varrho_{jk}(0) & \text{for } j \neq k. \end{cases} \tag{3.30}$$

Assuming that we know the common mean for λ_f's only,

$$\int_0^{\infty} ... \int^{\infty} p_0(\lambda_1, ..., \lambda_N)\lambda_j d\lambda_1 ... d\lambda_N = \Lambda_j, \tag{3.31}$$

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and assuming that all the eigenvalues are positive, we obtain the optimum distribution of the form

$$p_0(\lambda_1, \ldots, \lambda_N) = \prod_{j=1}^{N} \frac{1}{\Lambda_j} \exp\left(-\frac{\lambda_j}{\Lambda_j}\right) \tag{3.32}$$

and values for (3.21), received with (3.32) are

$$\langle \varrho_{jk}(t) \rangle_{p_0} = \begin{cases} \varrho_{jj}(0) & \text{for } j = k, \\ \left[\dfrac{1}{(1+\Lambda_j^2 t^2)(1+\Lambda_k^2 t^2)} + it \dfrac{\Lambda_k^2 - \Lambda_j^2}{(1+\Lambda_j^2 t^2)(1+\Lambda_k^2 t^2)}\right] \varrho_{jk}(0) & \text{for } j \neq k. \end{cases} \tag{3.33}$$

It is easy to see that, from (3.26), (3.30) and (3.33) for a sufficiently long time we have

$$\lim_{t\to\infty} \langle \varrho_{jk}(t) \rangle_{p_0} = \delta_{jk} \varrho_{jj}(0). \tag{3.34}$$

Denoting

$$\lim_{t\to\infty} \langle \varrho(t) \rangle_{p_0} = \varrho_\infty, \tag{3.35}$$

we get for the examples of the case b)

$$\varrho_\infty = A_\infty(0), \quad (A_\infty)_{ijkl} = \delta_{jk}\delta_{ik}\delta_{jl}. \tag{3.36}$$

(*cf.* Schlögl 1965, p. 301).

Case c. Let the joint probability distribution for $H \in \mathcal{H}_N'$ be denoted by $p(X, D)$ (*cf.* Eqs 3.2 and 3.4). Assuming that the statistical moments σ_j and Λ_j (being the analogues of (3.27, 3.28)) are known, we obtain the joint distribution as a product of the homogeneous distribution (3.7) over X(*i. e.* over O_N or U_N) and the Gaussian distribution (3.30) of the eigenvalues, *i. e.*

$$p_0(X, D) = p_0(X)\, p_0(D), p_0(X) = 1, p_0(D) = p_0(\lambda_1, \ldots \lambda_N). \tag{3.37}$$

For $H \in \mathcal{H}_N^r$, the optimum density operator matrix elements are

$$\langle \varrho_{jk}(t) \rangle_{p_0(D,R)} = (2\pi)^{-\frac{N}{2}} \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \ldots$$

$$\ldots \sum_{pqrs=1}^{N} \langle R_{jp} R_{qp} R_{rs} R_{ks} \rangle O_N \varrho_{qr}(0) \prod_{m=1}^{N} \frac{1}{\sigma_m} \exp\left[-\frac{1}{2\sigma_m^2} (\lambda_m - \Lambda_m)^2\right]. \tag{3.38}$$

After substituting (3.10) and (3.11) into (3.38) and carrying out the integration we obtain

$$\langle \varrho_{jj}(t) \rangle_{p_0(R,D)} = \frac{1}{N+2} \left\{ [2\varrho_{jj}(0) + \delta_{jj}] + \frac{2}{N-1} \left[\varrho_{jj}(0) - \frac{\delta_{jj}}{N}\right] \theta(t) \right\}, \tag{3.39}$$

$$\langle \varrho_{jk}(t) \rangle_{p_0(R,D)} = \frac{1}{N+2} [\varrho_{jk}(0) + \varrho_{kj}(0)] + \frac{2}{N(N-1)(N+2)} [(N+1)\varrho_{jk}(0) - \varrho_{kj}(0)] \theta(t)$$

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where

$$\theta(t) = \sum_{1 \leqslant p < q \leqslant N} (\cos \Lambda_p t)(\cos \Lambda_q t) \exp\left[-\frac{\sigma_p^2+\sigma_q^2}{2}\, t^2\right]. \tag{3.40}$$

For $H \in \mathcal{H}_N^c$ we obtain similar results

$$\langle \varrho_{jj}(t)\rangle_{p_0(U,D)} = \frac{1}{N+1}\left\{[\varrho_{jj}(0)+\varrho_{jj}] + \frac{2}{N(N-1)}\, N\varrho_{jj}(0) - \delta_{jj}]\theta(t)\right\}, \tag{3.41}$$

$$\langle \varrho_{jk}(t)\rangle_{p_0(U,D)} = \frac{1}{N+1}\, \varrho_{jk}(0)\left[1 + \frac{2}{N-1}\,\theta(t)\right]. \tag{3.42}$$

For a sufficiently long time our process becomes a stationary one

$$\lim_{t\to\infty} \langle \varrho_{jj}(t)\rangle_{p_0(R,D)} = \frac{1}{N+2}\,[2\varrho_{jj}(0)+\delta_{jj}], \tag{3.43}$$

$$\lim_{t\to\infty} \langle \varrho_{jk}(t)\rangle_{p_0(R,D)} = \frac{1}{N+2}\,[\varrho_{jk}(0)+\varrho_{kj}(0)], \tag{3.44}$$

and

$$\lim_{t\to\infty} \langle \varrho_{jj}(t)\rangle_{p_0(U,D)} = \frac{1}{N+1}\,[\varrho_{jj}(0)+\delta_{jj}], \tag{3.45}$$

$$\lim_{t\to\infty} \langle \varrho_{jk}(t)\rangle_{p_0(U,D)} = \frac{1}{N+1}\,\varrho_{jk}(0). \tag{3.46}$$

The formulae (3.43, 3.44), (or 3.45, 3.46) can be written in the form

$$\langle \varrho(t)\rangle_{p_0(R,D)} = A'_\infty \varrho(0) + a'I, \ (\varrho(0) \text{ diagonal}) \tag{3.47}$$

$$\langle \varrho(t)\rangle_{p_0(U,D)} = A''_\infty \varrho(0) + a''I \tag{3.48}$$

where

$$A' = \frac{2}{N+2}, \qquad A'' = \frac{1}{N+1}, \tag{3.49}$$

and

$$a = \frac{1-A_\infty}{N}, \tag{3.50}$$

respectively (*cf.* 3.16 and 3.17).

We can consider a more general example of the case c). Let us assume that, *e. g.*, for $H \in \mathcal{H}_N^r$ we know the statistical moments

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(H_{11}, \ldots, H_{NN}) H_{kl}\, d^n H = \mathcal{H}_{kl}^1, \tag{3.51}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(H_{11}, \ldots, H_{NN}) H_{kl}^2\, d^n H = \mathcal{H}_{kl}^2, \tag{3.52}$$

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where d^nH is given by (2.3). Maximizing the entropy

$$S[p] = -\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p(H_{11}, \dots, H_{NN}) \log p(H_{11}, \dots, H_{NN})\, d^nH, \tag{3.53}$$

we obtain the optimum probability distribution in the form

$$p_0(H_{11}, \dots, H_{NN}) = (2\pi)^{-\frac{N(N1)}{4}} \times \prod_{k \leqslant 1} \frac{1}{\sigma_{kl}} \exp\left\{-\left[\frac{1}{2\sigma_{kl}^2}(H_{kl}-\mathcal{H}_{kl}^1)^2\right]\right\}. \tag{3.54}$$

where

$$\sigma_{kl}^2 = \mathcal{H}_{kl}^2 - (\mathcal{H}_{kl}^1)^2. \tag{3.55}$$

Under special conditions (*cf.*, *e. g.*, Mehta 1960):

$$\sigma_{kl}^2 = \sigma^2,\ \mathcal{H}_{kl}^1 = 0, \text{ for } k \neq 1 \text{ and } k, l = 1, \dots, N,$$

$$\sigma_{kk}^2 = 2\sigma^2,\ \mathcal{H}_{kk}^1 = 0\ \ (k = 1, \dots, N), \tag{3.56}$$

from (3.54) we obtain the distribution

$$p_0(H_{11}, \dots, H_{NN}) = \exp\left(-\frac{\mathrm{Tr}\, H^2}{\sigma^2}\right), \tag{3.57}$$

used in the statistical theory of energy levels (*cf.*, *e. g.*, Wigner 1957, 1958, Porter 1960, Rozenzweig 1963, Dyson 1962a and others, see also Porter 1965). Derivation of the formula (3.57), based on the information theory methods, was made previously by Porter (1965) and Bronk (1965).

Now, we can calculate the optimum density operator (2.10) matrix elements, using the distribution (3.54), (or 3.57). But this encumbers enormous mathematical difficulties, and we leave this problem unsolved.

4. *The question of reversibility*

If we consider the evolution of a system described by a density operator, then the reversibility of this evolution is characterized by the behaviour of the Von Neumann entropy (*cf. e. g.*, Von Neumann 1955)

$$S[\varrho] = -\mathrm{Tr}\,(\varrho \log \varrho). \tag{4.1}$$

(We note that the entropy (4.1) describes our knowledge (information) about the state of a system and it is different from the entropy (2.9) which described our knowledge about the set $\mathcal{H}_N'$.)

The evolution is called irreversible if

$$S[\varrho(t)] > S[\varrho(0)], \tag{4.2}$$

whereas it is reversible when

$$S[\varrho(t)] = S[\varrho(0)] = \text{const.} \tag{4.3}$$

The entropy (4.1) remains constant during the evolution described by the Von Neumann equation (2.1), (*cf.*, *e. g.*, Schwegler 1965), *i. e.* the evolution is reversible. (Inversely, the Von Neumann equation (2.1) may be derived from the constancy of the entropy (4.1), *cf.* Schlögl 1965.)

Now we introduce the Von Neumann entropy for each mean density operator (2.6)

$$S[\langle\varrho(t)\rangle_p] = -\text{Tr}[\langle\varrho(t)\rangle_p \log \langle\varrho(t)\rangle_p], \tag{4.4}$$

and the optimum entropy $S[\langle\varrho(t)\rangle_{p_0}]$. Making use of the criterions (4.2) and (4.3) and utilizing the optimum entropy, we now examine the reversibility of the optimum evolution, *i. e.* that described by (2.1) and (2.10). We consider the cases a), b) and c) from Chap. 3.

Case a. According to the transformation (3.16) (with $\varrho(0)$ diagonal) the optimum entropy can be written in the form

$$S[\langle\varrho(t)\rangle_{p_0(R)}] = -\sum_i \left\{\left[A(t)\left(\varrho_{ii}(0) - \frac{\delta_{ii}}{N}\right) + \frac{\delta_{ii}}{N}\right] \times \right.$$

$$\left. \times \log\left[A(t)\left(\varrho_{ii}(0) - \frac{\delta_{ii}}{N}\right) + \frac{\delta_{ii}}{N}\right]\right\}, \tag{4.5}$$

where $A(t)$ is defined by (3.17). From (3.17) and (3.12) it follows that

$$\frac{2-N}{N+2} \leqslant A(t) \leqslant 1. \tag{4.6}$$

Now we make use of the well-known theorem about entropy, *cf.* Feinstein 1958, Chap. 2, which claims that for any sets $\{p_i\}$ and $\{q_i\}$, where $p_i > 0$, $q_i > 0$, ($i = 1, \ldots, N$), and

$$\sum_i p_i = 1\,, \sum_i q_i = 1\,, \tag{4.7}$$

the following inequality is satisfied

$$-\sum_i p_i \log p_i \leqslant -\sum_i p_i \log q_i\,. \tag{4.8}$$

Substytuting in (4.8) the term $\langle\varrho_{ii}(t)\rangle_{p_0(R)}$ instead of p_i, $\varrho_{ii}(0)$ instead of q_i (and assuming that $\langle\varrho_{ii}(t)\rangle_{p_0(R)}$ and $\varrho_{ii}(0)$ are greater than 0, $i = 1, \ldots, N$), we have

$$-\sum_i \langle\sigma_{ii}(t)\rangle_{p_0} \log \langle\varrho_{ii}(t)\rangle_{p_0} \leqslant -\sum_i \langle\varrho_{ii}(t)\rangle_{p_0} \log \varrho_{ii}(0) \tag{4.9}$$

and, inversely

$$-\sum_i \varrho_{ii}(0) \log \varrho_{ii}(0) \leqslant -\sum_i \varrho_{ii}(0) \log \langle\varrho_{ii}(t)\rangle_{p_0}. \tag{4.10}$$

Using the expression (4.5), we obtain that

$$S[\langle\varrho(t)\rangle_{p_0(R)}] \geqslant S[\varrho(0)]. \tag{4.11}$$

The expression for $A(t)$ is an almost periodic function (*cf.*, *e. g.*, Levitan 1953, Intoduction). The optimum entropy (4.5) being a function of $A(t)$, it is also an almost periodic function and hence we call it the almost periodic entropy. As is known, in some special cases an almost periodic function becemes a periedic one, *e.g.*, the expresion for $\Phi(t)$ (*cf.* 3.12)

$$\Phi(t) = \sum_{1\leqslant p<q\leqslant N} \cos \omega_{pq} t,\ \omega_{pq} = \lambda_p - \lambda_q,$$

becomes a periodic function, if the ω'_{pq}s form a set of commeasureable numbers. For example, if all ω'_{pq}s are equal to 1 or, generally speaking if

$$\underset{p,q}{A} \frac{\omega_{pq}}{2\pi} \tau = \alpha_{pq},\ \alpha_{pq} = 0,\ \pm 1,\ \pm 2, \ldots \tag{4.12}$$

then $\Phi(t)$ becomes a periodic function. In this case there exists at most a discrete set of moments τ, which are such that

$$A(0+\tau) = A(0) = 1, \tag{4.13}$$

and, from (3.15), for $A(\tau)$ equal to 1,

$$S[\langle\varrho(\tau)\rangle_{p_0}] = S[\varrho(0)]. \tag{4.14}$$

Because of (4.11) and (4.14), we obtain a "pulsating" entropy and we call the optimum evolution in this case an almost irreversible evolution (such an evolution is irreversible except for aset of discrete moments of time). If ω'_{pq}s are linearly independent (uncommeasureable), then the entropy (4.5) is always an almost periodic function. For an almost periodic function $f(x)$ there is defined an ε, T almost period τ, such that

$$|f(x+\tau) - f(x)| < \varepsilon \tag{4.15}$$

for all $|x| < T$. Here we may introduce an ε, T almost period τ for the almost periodic entropy (4.5), *i. e.*, *e. g.*,

$$|S[\langle\varrho(\tau)\rangle_{p_0}] - S[\varrho(0)]| < \varepsilon. \tag{4.16}$$

The evolution is now irreversible (at most ε-reversible, according to (4.16)). The magnitude of our ε, T almost period is comparable with the magnitude of the Poincare cycle in classical physics (*cf.*, *e. g.* Kac 1957, Chap. 3).

All our conclusions remain valid for the mean density operator $\langle\varrho(t)\rangle_{p_0(U)}$.

Case b. We learned that for a sufficiently long time (*cf.* 3.34)

$$\lim_{t\to\infty} \langle\varrho_{jk}(t)\rangle_{p_0}(\lambda_1, \ldots, \lambda_N) = \delta_{jk}\varrho_{jj}(0),\ (j, k, 1, \ldots N).$$

We shall compare the entropy $\underset{t\to\infty}{S\langle\varrho(t)\rangle_{p_0}}$ and $S[\varrho(0)]$. For $\varrho(0)$ one can find a representation r_{ik} such that

$$\varrho_{mn}(0) = \sum_p r_{mp}\varrho_p^D r^*_{np}, \tag{4.17}$$

where ϱ_p^D are the eigenvalues of $\varrho(0)$. In particular, the diagonal elements of $\varrho(0)$ are

$$\varrho_{ii}(0) = \sum_p |r_{ip}|^2 \varrho_p^D. \tag{4.18}$$

In this representation the initial entropy is equal to

$$S[\varrho(0)] = S(0) = -\sum_p \varrho_p^D \log \varrho_p^D. \tag{4.19}$$

For $t \to \infty$ the entropy tends to

$$\underset{t\to\infty}{S\langle\varrho(t)\rangle_{p_0}} = -\sum_i \varrho_{ii}(0) \log \varrho_{ii}(0). \tag{4.20}$$

Because of (4.18) we can use the other well-known theorem about entropy (cf. Feinstein 1958, Chap. 2). This theorem says that for two arbitrary sets, $\{p_i\}$ and $\{a_{ij}\}$, of numbers satisfying the conditions

$$\sum_i a_{ij} = \sum_j a_{ij} = 1, \sum_i p_i = 1, p_i > 0, (i = 1, \ldots, N) \tag{4.21}$$

the inequality

$$-\sum_j p_i' \log p_i' \geqslant -\sum_i p_i \log p_i, \tag{4.22}$$

is satisfied, where

$$p_i' = \sum_{j=1}^N a_{ij} p_j. \tag{4.23}$$

In our case we replace:

$$a_{ij} \to |r_{ip}|^2$$

$$p_j \to \varrho_p^D$$

$$p_i' \to \varrho_{ii}(0),$$

and obtain

$$\underset{t\to\infty}{S[\langle\varrho(t)\rangle_{p_0}(\lambda_1, \ldots, \lambda_N)]} \geqslant S(0). \tag{4.25}$$

Hence, if $\varrho(0)$ has any off-diagonal elements, the optimum evolution is irreversible (the sign $>$ in (4.25)). Otherwise, it is reversible (the sign $=$ in (4.25)).

Case c. Here we also define the limiting entropy

$$\underset{t\to\infty}{S[\langle\varrho(t)\rangle]} = -\sum_i \varrho_{ii}(t) \log \varrho_{ii}(t), \tag{4.26}$$

utilizing the formulae (3.43), (3.44), or (3.45), (3.46), and assume that $\varrho(0)$ is given in the diagonal form. The initial entropy is

$$S(0) = -\sum_i \varrho_{ii}(0) \log \varrho_{ii}(0). \tag{4.27}$$

In analogy to the case a), we use the same theorem (4.8) about entropy and obtain

$$\underset{t\to\infty}{S[\langle\varrho(t)\rangle]} \geqslant S(0). \tag{4.28}$$

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Here the evolution is also irreversible, except for the case when

$$\varrho_{jk}(0) = N^{-1}\,\delta_{jk}. \tag{4.29}$$

The last conclusion is well-known: for $\varrho_{jk}(0)$ given in the form (4.29), the Von Neumann entropy (4.1) attains its maximum value, which is equal to log N, so that this entropy cannot increase (*cf.* also Schlögl 1965).

5. *Conclusions*

In the present paper we stated that the time evolution of a system with a stochastic, time independent Hamiltonian is, in general, irreversible. The irreversibility is strictly connected with the randomness of the Hamiltonian eigenvalues (cases *b*, and *c*): in the case *a*, there appears a special type od irreversibility, which we call the almost irreversibility. The evolution of a system with a stochastic Hamiltonian can be considered as a model of a "quantum random walk".

In all considered cases the optimum density operator and entropy depend only on the absolute value of time, *cf.* (3.10), (3.11), (3.14), (3.15), (3.26), (3.30), (3.33), (3.39), (3.41), (3.42) and (4.4), so they are symmetrical with respect to the initial moment $t = 0$.

We hope that our remarks may throw a new light on the problem of the reversibility and irreversibility of physical processes.

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APPENDIX I

Averaging over the orthogonal group O_N

We need to find the value of the expression (3.9)

$$\langle R_{jp}R_{qp}R_{rs}R_{ks}\rangle_{O_N} = \int\limits_{O_N} R_{jp}R_{qp}R_{rs}R_{ks}\mu(dR),$$

where $\mu\,(dR)$ is the invariant measure of the orthogonal group O_N. Although such expressions were calculated previously by Ullah and Porter (1963), we will briefly outline another method of calculation which is also applicable to the case of unitary groups. For any function $F(x_i) = F(x_1, \ldots x_N)$, where x_i are the polar co-ordinates in an N-dimensional Euclidean space, we have (*cf.*, *e. g.*, Weyl 1940, Chap. 7, also Gelfand 1958, Chap. 1):

$$\int\limits_{O_N} F(R\vec{x})\mu(dR) = \frac{1}{S_N}\int\limits_{S_N} F(\vec{x})d\sigma, \tag{AI.1}$$

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where

$$\int_{O_N} \mu(dR) = 1,\ R \in O_N,\ x = (x_1, \ldots, x_N), \tag{AI.2}$$

and S_N is the surface of the N-dimensional sphere.

We choose

$$F(x_i) = x_i x_j x_k x_l, \tag{AI.3}$$

and obtain, from (AI.1), for any $R \in O_N$

$$\sum_{pqrs=1}^{N} x_p x_q x_r x_s \int_{O_N} R_{ip} R_{jq} R_{kr} R_{ls} \mu(dR) = \frac{1}{S_N} \int_{S_N} x_i x_j x_k x_l d\sigma. \tag{AI.4}$$

We now use the normalization of x_j's

$$\sum_{j=1}^{N} x_j^2 = 1, \tag{AI.5}$$

and the orthogonality of R_{ik}'s

$$\sum_{m=1}^{N} R_{mn} R_{pn} = \delta_{mp}. \tag{AI.6}$$

Substituting in (A1.4) x_p

$$x_p = \delta_{mp} \tag{AI.7}$$

and, then

$$x_p = \sigma_{pm}\alpha + \delta_{pn}\beta,\ m \neq n,\ \alpha^2 + \beta^2 = 1, \tag{AI.8}$$

we obtain the formulae

$$\langle R_{ij}^4 \rangle_{O_N} = \frac{3}{N(N+2)}$$

$$i \neq j \quad \langle R_{ip}^2 R_{jq}^2 \rangle_{O_N} = \frac{1}{N(N+2)}$$

$$\begin{matrix} i \neq j \\ p \neq q \end{matrix} \quad \langle R_{ip}^2 R_{jq}^2 \rangle_{O_N} = \frac{N+1}{N(N-1)(N+2)}$$

$$\begin{matrix} i \neq r \\ p \neq q \end{matrix} \quad \langle R_{ip} R_{rp} R_{rq} R_{iq} \rangle_{O_N} = -\frac{1}{N(N-1)(N+2)}. \tag{AI.9}$$

APPENDIX II

Averaging over the unitary group U_N

Now we calculate the expression

$$\langle U_{iq} U_{rq}^* U_{sp} U_{kp}^* \rangle_{U_N} = \int_{U_N} U_{jq} U_{rq}^* U_{sp} U_{kp}^* \, \mu(dU). \tag{A.II.1}$$

We use the method of Appendix I. In analogy to (AI.1.) we have

$$\int\limits_{U_N} F(U\vec{z})\mu(dU) = \frac{1}{S'_{2N}} \int\limits_{S'_{2N}} F(\vec{z})d\sigma, \tag{AII.2}$$

where $\vec{z}$ is an N-dimensional complex vector, whose square absolute value is

$$|\vec{z}|^2 = \sum_{i=1}^{N} z_i z_i^*, \tag{AII.3}$$

and S'_{2N} is the surface of the $2N$-dimensional sphere (or some part of it).

We choose $F(\vec{z})$ in the following form

$$F(\vec{z}) = z_i z_j^* z_k z_l^*. \tag{AII.4}$$

Substituting (AII.4) into (AII.2) with

$$\vec{z}_k = \delta_{kt} e^{i\varphi} \tag{AII.5}$$

or

$$z_k = \delta_{kt}\alpha + \delta_{kw}\beta,\ |\alpha|^2 + |\beta|^2 = 1,\ t \neq w, \tag{AII.6}$$

and using the orthogonality of U'_{jk}s

$$\sum_{m=1}^{N} U_{im} U_{jm}^* = \delta_{ij}, \tag{AII.7}$$

we finaly obtain the formulae

$$\begin{aligned} &\langle |U_{ij}^4| \rangle_{U_N} = \frac{2}{N(N+1)}, \\ &i \neq j\ \langle |U_{im}|^2 |U_{jm}|^2 \rangle_{U_N} = \frac{1}{N(N+1)}, \\ &\begin{matrix} i \neq j \\ m \neq n \end{matrix}\ \langle |U_{im}|^2 |U_{jn}|^2 \rangle_{U_N} = \frac{1}{(N-1)(N+1)}, \\ &\begin{matrix} i \neq r \\ p \neq q \end{matrix}\ \langle U_{ip} U_{rp}^* U_{rq} U_{iq}^* \rangle_{U_N} = -\frac{1}{N(N-1)(N+1)}. \end{aligned} \tag{AII.8}$$

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