THE INITIAL VALUE PROBLEM FOR A PLASMA IN AN EXTERNAL FIELD

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The longitudinal plasma oscillations in an external electric field are investigated by means of numerical solutions and an approximation technique. For weak external fields the phase integral approximation and multitimescales perturbation techniques are used. The approximate solutions in the form of damped oscillations with time-dependent frequencies are found. The mechanism of instability growth is discussed.

I. Introduction

The problem of oscillations of electrons imbedded in a continuous neutralizing medium in the presence of an external field may be soved analitically. The only effect of the external field is the acceleration of electrons, and a time-dependent phase factor appears.

The physical picture is changed if we take into account the two-component plasma (electrons and ions). In this case the velocity distribution function evolve into two-humped distribution function, and two-stream instability may occur. This problem has been previously studied by Fried, Gell-Mann, Jackson and Wyld [1] in the frame of the linear and quasilinear theory. In linear theory they reduce the problem to a Volterra type integral equation, however they use a rather rough approximation and their discussion is only qualitative.

We give here a more detailed discussion of this case by means of numerical solutions and by the approximation techniques. The multitimescales perturbation theory, valid for weak external fields, and the phase integral method are used. Results obtained for short times are in good agreement with the numerical calculations. All numerical calculations have been performed on the GIER computer.

II. Basic equations

We consider a gas consisting of electrons and ions which an uniform external electric field is acting. The evolution of distribution functions of both species is described by linea-

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rized Vlasov kinetic equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - em^{-1} \boldsymbol{E}_e \cdot \frac{\partial f}{\partial \boldsymbol{v}} - em^{-1} \boldsymbol{E} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0$$

$$\frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F}{\partial \boldsymbol{x}} + eM^{-1} \boldsymbol{E}_e \cdot \frac{\partial F}{\partial \boldsymbol{v}} + eM^{-1} \boldsymbol{E} \cdot \frac{\partial F_0}{\partial \boldsymbol{v}} = 0$$

$$\frac{\partial f_0}{\partial t} - em^{-1} \boldsymbol{E}_e \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0$$

$$\frac{\partial F_0}{\partial t} + eM^{-1} \boldsymbol{E}_e \cdot \frac{\partial F_0}{\partial \boldsymbol{v}} = 0 \tag{2.1}$$

where

$$\frac{\partial}{\partial \boldsymbol{x}} \cdot \boldsymbol{E} = 4\pi e \int d\boldsymbol{v}\,(F-f) \tag{2.2}$$

f_0 and F_0 are velocity distribution functions, f and F are inhomogeneity factors, and m and M are the masses of electrons and ions, respectively, e is electron charge (absolute value), and E_e is the external, uniform, time-dependent electric field.

The solution for the velocity distribution functions may be written as follows:

$$f_0(\boldsymbol{v}, t) = n_0 \varphi(\boldsymbol{u})$$

$$F_0(\boldsymbol{v}, t) = n_0 \Phi(\boldsymbol{w}) \tag{2.3}$$

where

$$\boldsymbol{u} = \boldsymbol{v} + em^{-1} \int_0^t \boldsymbol{E}_e(t')dt'$$

$$\boldsymbol{w} = \boldsymbol{v} - eM^{-1} \int_0^t \boldsymbol{E}_e(t')dt' \tag{2.4}$$

n_0 is the mean density, φ and Φ are arbitrary positive functions which satisfy the condition

$$\int \varphi(\boldsymbol{u})d\boldsymbol{u} = \int \Phi(\boldsymbol{w})d\boldsymbol{w} = 1 \tag{2.5}$$

Let us now take the Fourier transformation with respect to $\boldsymbol{x}$-dependence. The equations (2.1) and (2.2) become

$$\frac{\partial f_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k} \cdot \boldsymbol{v} f_{\boldsymbol{k}} - em^{-1} \boldsymbol{E}_e \cdot \frac{\partial f_{\boldsymbol{k}}}{\partial \boldsymbol{v}} - en_0 m^{-1} \boldsymbol{E}_{\boldsymbol{k}} \cdot \frac{\partial \varphi(\boldsymbol{u})}{\partial \boldsymbol{v}} = 0$$

$$\frac{\partial F_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k} \cdot \boldsymbol{v} F_{\boldsymbol{k}} + eM^{-1} \boldsymbol{E} \cdot \frac{\partial F_{\boldsymbol{k}}}{\partial \boldsymbol{v}} + en_0 M^{-1} \boldsymbol{E}_{\boldsymbol{k}} \cdot \frac{\partial \Phi(\boldsymbol{w})}{\partial \boldsymbol{v}} = 0 \tag{2.6}$$

and

$$\boldsymbol{E}_{\boldsymbol{k}} = -4\pi e i k^{-2} \boldsymbol{k} \int d\boldsymbol{v}(F_{\boldsymbol{k}} - f_{\boldsymbol{k}}) \tag{2.7}$$

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This system can be reduced [1] to the Volterra type integral equation for the electric field $\boldsymbol{E}_k$,

$$\boldsymbol{E}_k(t)+\omega_p^2\int_0^t dt'\boldsymbol{E}_k(t')[t-t']\{\tilde{\varphi}[\boldsymbol{k}(t-t')]\exp[i\boldsymbol{k}(\boldsymbol{\eta}-\boldsymbol{\eta}')]+$$

$$+mM^{-1}\tilde{\Phi}[\boldsymbol{k}(t-t')]\exp[-i\boldsymbol{k}(\boldsymbol{\eta}-\boldsymbol{\eta}')m/M]\}$$

$$=4\pi eik^{-2}\boldsymbol{k}\{\exp(i\boldsymbol{k}\boldsymbol{\eta})\tilde{f}_k(\boldsymbol{k}t,0)-\exp(im\boldsymbol{k}\boldsymbol{\eta}/M)\tilde{F}_k(\boldsymbol{k}t,0)\},\tag{2.8}$$

where

$$\omega_p^2=4\pi n_0e^2m^{-1}\tag{2.9}$$

is the electron plasma frequency and the tilde over f, F, f and F denotes the Fourier transform in velocity space,

$$\tilde{F}(\mathbf{s},t)=\int_{-\infty}^{\infty}d\boldsymbol{v}\exp(-i\boldsymbol{v}\mathbf{s})F(\boldsymbol{v},t)\tag{2.10}$$

where

$$\boldsymbol{\eta}=e/m\int_0^t(t-t'')\boldsymbol{E}_e(t'')dt'';\quad \boldsymbol{\eta}'=\boldsymbol{\eta}(t')\tag{2.11}$$

$\boldsymbol{E}$ is parallel to $\boldsymbol{k}$ and we can consider this problem to be one dimensional. Equation (2.8) has been studied numerically on the GIER computer.

III. Phase integral approximation

For plasma oscillations in the absence of any external field we can find the solution of Eqs (2.1) in the form of growing (unstable case) or damped (stable case) oscillations. The frequencies of these oscillations are obtained by solving the dispersion equation

$$1-\omega_p^2k^{-2}\int\frac{du}{u-w}\left\{\frac{\partial\varphi}{\partial u}+\frac{m}{M}\frac{\partial\Phi}{\partial u}\right\}\tag{3.1}$$

In our case we are looking for a solution of Eqs (2.1) close to the above mentioned Landau type solution.

Let us perform the change of variables in the equations (2.6). In the first of these

$$\boldsymbol{v}\to\boldsymbol{u},\quad t\to t;$$

and in the second

$$\boldsymbol{v}\to\boldsymbol{w},\quad t\to t;$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are defined by Eqs (2.4). The equations (2.1) can be written as follows:

$$\frac{\partial f_k(\boldsymbol{u},t)}{\partial t}+i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{\xi})f_k(\boldsymbol{u},t)-en_0m^{-1}\boldsymbol{E}_k(t)\frac{\partial\varphi(\boldsymbol{u})}{\partial\boldsymbol{u}}=0$$

$$\frac{\partial F_k(\boldsymbol{w},t)}{\partial t}+i\boldsymbol{k}(\boldsymbol{w}+mM^{-1}\boldsymbol{\xi})F_k(\boldsymbol{w},t)+en_0M^{-1}\boldsymbol{E}_k(t)\frac{\partial\Phi(\boldsymbol{w})}{\partial\boldsymbol{w}}=0\tag{3.2}$$

and

$$\boldsymbol{E}_k(t)=-4\pi eik^{-2}\boldsymbol{k}\{\int d\boldsymbol{w}F_k(\boldsymbol{w},t)-\int d\boldsymbol{u}f_k(\boldsymbol{u},t)\}\tag{3.3}$$

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where

$$\boldsymbol{\xi} = em^{-1} \int_0^t dt' \boldsymbol{E}_e(t') \tag{3.4}$$

Introducing the "barring" operation,

$$\bar{F}(u) = \int d\boldsymbol{u}' F(\boldsymbol{u}') \delta(u - \boldsymbol{k} \cdot \boldsymbol{u}'/k)$$
$$u = \boldsymbol{u} \cdot \boldsymbol{k}/k \tag{3.5}$$

and taking into account that

$$\boldsymbol{E}_k \sim \boldsymbol{k}$$

and

$$\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{u}} = k \frac{\partial}{\partial u} \tag{3.6}$$

we obtain

$$\frac{\partial \bar{f}_k(u, t)}{\partial t} + ik(u - \xi) \bar{f}_k(u, t) - en_0 m^{-1} E_k \frac{\partial \bar{\varphi}(u)}{\partial u} = 0$$

$$\frac{\partial \bar{F}_k(w, t)}{\partial t} + ik(w - mM^{-1}\xi) \bar{F}_k(w, t) + en_0 M^{-1} E_k \frac{\partial \bar{\Phi}(w)}{\partial w} = 0 \tag{3.7}$$

and

$$E_k(t) = -4\pi eik^{-1} \{ \int_{-\infty}^{\infty} d\omega \bar{F}_k(w, t) - \int_{-\infty}^{\infty} du \bar{f}_k(u, t) \} \tag{3.8}$$

where

$$\xi = k^{-1} \boldsymbol{k} \cdot \boldsymbol{\xi} \tag{3.9}$$

We are looking for the solutions in the following form:

$$f_k(u, t) = \psi_k(u, t) \exp \{-ik \int_0^t dt' \gamma(t')\} \tag{3.10a}$$

and

$$F_k(u, t) = \Psi_k(u, t) \exp \{-ik \int_0^t dt' \gamma(t')\} \tag{3.10b}$$

where $\psi_k(u, t)$ and $\Psi_k(u, t)$ are slowly warying functions of time. The equations for amplitudes $\psi_k(u, t)$ and $\Psi_k(u, t)$ can be written in the following form

$$ik(u - \xi - \gamma) \psi_k(u, t) - en_0 m^{-1} \mathcal{E}_k(t) \frac{\partial \bar{\varphi}}{\partial u} = 0$$

$$ik(u + mM^{-1}\xi - \gamma) \Psi_k(u, t) + en_0 M^{-1} \mathcal{E}_k(t) \frac{\partial \bar{\Phi}}{\partial u} = 0 \tag{3.11}$$

and equation for self-consistent in the form

$$\mathcal{E}_k(t) = -4\pi eik^{-1} \int_{-\infty}^{\infty} du \{\Psi_k(u, t) - \psi_k(u, t)\} \tag{3.12}$$

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where the time derivatives are neglected. Equations (3.11) and (3.12) are very similar to the equations for the normal modes in the Case-van Kampen [2—4] treatment of the Vlasov equation for plasma in the absence of external field. The dispersion relation for determination of $\gamma(t)$ can be immediately obtained and has the form

$$\varepsilon_k^+(\gamma, t) \equiv 1-\omega_p^2 k^{-2} \int du \left\{ \frac{\frac{\partial \bar{\varphi}(u)}{\partial u}}{u-\xi-\gamma} + \frac{m}{M} \frac{\frac{\partial \Phi(u)}{\partial u}}{u+\frac{m}{M}\xi-\gamma} \right\} = 0 \tag{3.13}$$

where ω_p is defined by Eq. (2.9). This dispersion relation is different than in the papers [2—4] by the presence of the time dependent function $\xi(t)$ in the denominators.

In the case of unstable plasma we can find the discrete modes and these should be the proper solution of our problem. In the case of stable plasma the discrete van Kampen-Case modes do not exist, but we can find the generalized discrete modes introduced by Trocheris [5]. This modes can be obtained by finding the zeros of an analytical continuation of the dispersion relation to the lower half-plane. For of plasma without external fields these modes are identical to the Landau solutions. By solving a dispersion equation we obtain a set of functions $\gamma_j(t)$. From the fact that

$$\xi(t)\big|_{t=0} = 0$$

we observe that

$$\gamma_j(t)\big|_{t=0} = \gamma_j \tag{3.14}$$

where γ_j are the respective Landau poles for plasma in absence of external field.

This give us the clear physical interpretation of the solutions. The plasma oscillates with its natural frequencies which change in time due to the change of the homogeneous components of plasma distribution functions, generated by the external field.

To complete the solution we have to determine the expansion coefficients for the modes. This can be done by solving the conjugate Vlasov equation and using the orthogonality theorem, but as we know, this is equivalent to Landau procedure. We can simply write

$$A^s = -4\pi eik^{-1} \,\mathrm{Re} \left\{ \frac{\lambda(\gamma)}{\varepsilon_k^+(\gamma, t)} \right\}_{\gamma=\gamma_s(t)} \tag{3.15}$$

where A^s is the expansion coefficient, and

$$\lambda(z) = \int du \frac{1}{u-z} \{\bar{f}_k^{(0)}(u) - \bar{F}_k^{(0)}(u)\} \tag{3.16}$$

where we assume the initial condition in the form

$$\begin{aligned} \bar{F}_k(u, 0) &= \bar{F}_k^{(0)}(u) \\ \bar{f}_k(u, 0) &= \bar{f}_k^{(0)}(u) \end{aligned} \tag{3.17}$$

The solution for self-consistent field can be written as follows

$$E_k(t) = \sum_s A^s \exp\{-ik \int_0^t dt' \gamma_s(t')\} \tag{3.18}$$

One can improve this approximation by introducing a time-dependent correction factor $\delta_1(t)$ by means of the relation

$$E_k(t) \to \delta_1(t) E_k(t) \tag{3.19}$$

with the initial condition

$$\delta_1(0) = 1 \tag{3.20}$$

It is clear that it is impossible to satisfy the whole set of equations by such simple substitution, but one can satisfy the reduced equation for a self-consistent electric field. This equation reduce to a firts-order differential equation for $\delta_1(t)$.

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IV. The multitimescale perturbation technique

To find the approximate solution of Eqs (2.6) one may use also the multiscale perturbation which is amodification of the Poincaré-Bogolyuboff [6] theory. We will assume here that the external field is of the order of λ, where λ is a small parameter. We assume also that all the functions depend on time in different time scales.

$$\bar{f}_k = \sum_{n=0}^{\infty} \lambda^n \bar{f}_k^{(n)}(v, t_0, \lambda t_1, \ldots, \lambda^s t_s, \ldots, \ldots),$$

$$\bar{F}_k = \sum_{n=0}^{\infty} \lambda^n \bar{F}_k^{(n)}(v, t_0, \lambda t_1, \ldots, \lambda^s t_s, \ldots, \ldots)$$

$$E_k = \sum_{n=0}^{\infty} \lambda^n E_k^{(n)}(t_0, \lambda t_1, \ldots, \lambda^s t_s, \ldots, \ldots) \tag{4.1}$$

where we have

$$\frac{dt_s}{dt} = 1 \qquad (s = 0, 1, \ldots, n, \ldots) \tag{4.2}$$

with the initial conditions

$$t_s(0) = 1 \tag{4.3}$$

Using this expansion we can express the time derivatives in the form

$$\frac{\partial f}{\partial t} = \sum_{n=0}^{\infty} \lambda^n \frac{\partial}{\partial(\lambda^n t_n)} f(t_0, \lambda t_1, \ldots, \lambda^s t_s, \ldots) \tag{4.4}$$

Substituting this expansion into Eqs (2.1) and taking into account (3.5) and (4.4) we obtain the set of equations

$$\frac{\partial}{\partial t_0} \bar{f}_k^{(s)} + ikv\bar{f}_k^{(s)} - \frac{en_0}{m} E_k^{(s)} \frac{\partial \bar{\varphi}(u)}{\partial v} = -\frac{\partial \bar{f}_k^{(s-1)}}{\partial(\lambda t_1)} + \frac{e}{m} E_e \frac{\partial \bar{f}_k^{(s-1)}}{\partial v} - \sum_{l=2}^{s} \frac{\partial \bar{f}_k^{(s-l)}}{\partial(\lambda^l t_l)}$$

$$\frac{\partial}{\partial t_0} \bar{F}_k^{(s)} + ikv\bar{F}_k^{(s)} + \frac{en_0}{M} E_k^{(s)} \frac{\partial \Phi(w)}{\partial v} = -\frac{\partial F_k^{(s-1)}}{\partial(\lambda t_1)} - \frac{e}{M} E_e \frac{\partial F_k^{(s-1)}}{\partial v} - \sum_{l=2}^{s} \frac{\partial F_k^{(s-l)}}{\partial(\lambda^l t_l)} \tag{4.5}$$

where the following notation has been used

$$u = v + \xi(\lambda t_1)$$
$$w = v - (m/M)\xi(\lambda t_1). \tag{4.6}$$

and the equation for self-consistent field

$$E_k^{(s)} = -4\pi e k^{-1} i \int dv \, \{\bar{F}_k^{(s)}(v) - \bar{f}_k^{(s)}(v)\} \tag{4.7}$$

We will take the nonvanishing initial condition only for the functions with $s = 0$.

Since our equations are partial differential equations with varying ceofficients we cannot use the standart assumption that the left-hand sides of Eqs (4.5) are identically zero. With this additional step one obtaines contradiction and inconsistency. From the form of the equations it is clear that one can limit the number of time scales to two: the first one is a short time scale, and the second a long time scale.

The lowest order equations have the form

$$\frac{\partial}{\partial t_0} \bar{f}_k^{(0)} + ikv\bar{f}_k^{(0)} - \frac{en_0}{m} E_k^{(0)} \frac{\partial \varphi(u)}{\partial v} = 0$$
$$\frac{\partial}{\partial t_0} \bar{F}_k^{(0)} + ikv\bar{F}_k^0 + \frac{en_0}{M} E_k^{(0)} \frac{\partial \Phi(w)}{\partial v} = 0 \tag{4.8}$$

and the higher order equations have the following form

$$\frac{\partial \bar{f}_k^{(s)}}{\partial t_0} + ikv\bar{f}_k^{(s)} - \frac{en_0}{m} E_k^{(s)} \frac{\partial \varphi(u)}{\partial t_0} = -\frac{\partial \bar{f}_k^{(s-1)}}{\partial \lambda t_1} + \frac{e}{m} \frac{\partial \bar{f}_k^{(s-1)}}{\partial v} E_e = g_k^{(s)}$$
$$\frac{\partial \bar{F}_k^{(s)}}{\partial t_0} + ikv\bar{F}_k^{(s)} - \frac{en_0}{M} E_k^{(s)} \frac{\partial \bar{\Phi}(w)}{\partial t_0} = -\frac{\partial \bar{F}_k^{(s-1)}}{\partial \lambda t_1} + \frac{e}{M} \frac{\partial \bar{F}_k^{(s-1)}}{\partial v} E_e \equiv G_k^{(s)} \tag{4.9}$$

for $s > 1$. Performing the Laplace transformation with respect to the time t_0,

$$\hat{f}_k^{(s)}(u, z) = \int_0^\infty dt_0 e^{ikzt_0} \bar{f}_k^{(s)}(u, t_0, \lambda t_1)$$
$$\hat{F}_k^{(s)}(u, z) = \int_0^\infty dt_0 e^{ikzt_0} \bar{F}_k^{(s)}(u, t_0, \lambda t_1) \tag{4.10}$$

and changing the variables in the equation for $\bar{f}_k^{(s)}$ from (v, t_0, t_1) to $(u, t_0\, t_1)$, and in the equation for $\bar{F}_k^{(s)}$ from (v, t_0, t_1) to (w, t_0, t_1) we can rewrite the equations (4.8) and (4.9) uniformly

$$ik(u - \xi - z)\hat{f}_k^{(s)}(u, z, t_1) - \frac{en_0}{m} E_k^{(s)}(z, t_1) \frac{\partial \bar{\varphi}}{\partial u} = g_k^{(s)}$$
$$ik\left(w + \frac{m}{M}\,\xi - z\right) \hat{F}_k^{(s)}(u, z, t_1) + \frac{en_0}{M} E_k^{(s)}(z, t_1) \frac{\partial \bar{\Phi}}{\partial w} = \hat{G}_k^{(s)} \tag{4.11}$$

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where

$$\hat{g}_k^{(0)} = \bar{f}_k^{(0)}(u-\xi)$$

$$\hat{G}_k^0 = \bar{F}_k^{(0)}\left(w + \frac{m}{M}\,\xi\right) \tag{4.12}$$

are the initial conditions. $\hat{G}_k^{(s)}$ and $\hat{g}_k^{(s)}(s > 0)$ are the Laplace transforms of $G_k^{(s)}$ and $g_k^{(s)}$. The solutions of these equations have the well-known form

$$E_k^{(s)}(z, t_1) = 4\pi e k^{-1} i \, \frac{\psi_k^{(s)}(z+\xi) - \psi_1^{(s)}\left(z - \frac{m}{M}\,\xi\right)}{\varepsilon_k^+(z, t_1)} \tag{4.13}$$

where $\varepsilon_k^+(z, t_1)$ is determined by formula (3.12) with z instead of γ, and

$$\psi_1^{(s)}(z) = -ik^{-1} \int\limits_{-\infty}^{\infty} \frac{du}{u-z}\, g_k^{(s)}(u)$$

$$\Psi_1^{(s)}(z) = -ik^{-1} \int\limits_{-\infty}^{\infty} \frac{du}{u-z}\, G_k^{(s)}(u) \tag{4.14}$$

This procedure gives us the solution in the arbitrary order in λ. The physical picture described by the lowest order solutions is very clear and very close to that obtained by the phase integral method. If one writes

$$E_k^{(0)}(t) = 2ek^{-1} \int\limits_c \frac{\exp(-ikzt)}{\varepsilon_k^+(z, t)} \left\{\psi_1^{(0)}(z+\xi) + \Psi_1^{(0)}\left(z - \frac{m}{M}\,\xi\right)\right\} \tag{4.15}$$

and takes the contour integral (using Landau arguments), one can obtain the solution in the form

$$E_k^{(0)}(t) = \sum_s A^s(t) \exp\left[-ik\gamma_s(t)t\right] \tag{4.16}$$

where we have put

$$t_0 = t_1 = t \tag{4.17}$$

and where $\gamma_s(t)$ is determined from the dispersion relation (3.13).

V. Model calculations

In order to perform the numerical calculations we shall use a model of distribution functions. We choose the distribution function introduced by Jackson [7].

$$\varphi(\boldsymbol{v}) = \frac{a}{\pi^2}\, \frac{1}{[(\boldsymbol{v}-\boldsymbol{u})^2 + a^2]^2} \tag{5.1}$$

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where $\boldsymbol{u}$ has the meaning of mean velocity and a^2 is the temperature divided by mass. After performing the integration over the components of $\boldsymbol{v}$ perpendicular $\boldsymbol{k}$ we obtain

$$\bar{\varphi}(v) = \frac{a_1}{\pi} \frac{1}{(v-u_1)^2+a_1^2}$$

$$\bar{\Phi}(v) = \frac{a_3}{\pi} \frac{1}{(v-u_3)^2+a_3^2} \tag{5.2}$$

By simple contour integrations we obtain

$$\chi(z) \equiv \int \frac{dv}{v-z} \frac{\partial \bar{\varphi}}{\partial v} = \frac{1}{(ia_1+2z-u_1)^2}$$

$$X(z) \equiv \int \frac{dv}{v-z} \frac{\partial \Phi}{\partial v} = \frac{1}{(ia_3+z-u_3)^2} \tag{5.3}$$

The dielectric constant $\varepsilon_k^+(z, t)$ can be written in the terms of χ, X functions,

$$\varepsilon_k^+(z, t) = 1 - \frac{\omega_p^2}{k^2} \left\{\chi(z+\xi) + \frac{m}{M} X\left(z - \frac{m}{M} \xi\right)\right\} \tag{5.4}$$

and we obtain the dispersion relation in the form of fourth-order algebraic equation,

$$\varepsilon_k^+(z, t) = 1 - \frac{\omega_p^2}{k^2} \left\{\frac{1}{(z+\xi-u_1+ia_1)^2} + \frac{m}{M} \frac{1}{\left(z - \frac{m}{M} \xi-u_3+ia_3)^2\right)}\right\} = 0 \tag{5.5}$$

This can be rewritten in the form

$$Z^4-2(\alpha_1+\alpha_3)z^3 + \left[\alpha_1^2+\alpha_3^2+4\alpha_1\alpha_3 - \frac{\omega_p^2}{k^2}\left(1 + \frac{m}{M}\right)\right] z^2-$$

$$-2\left[\alpha_1\alpha_3(\alpha_1+\alpha_3) - \frac{\omega_p^2}{k^2}\left(\alpha_3 + \frac{m}{M}\alpha_1\right)\right] z+$$

$$+\alpha_1^2\alpha_3^2 - \frac{\omega_p^2}{k_2}\left(\alpha_3^2 + \frac{m}{M}\alpha_1^2\right) = 0 \tag{5.6}$$

where

$$\alpha_1 = u_1-\xi-ia_1; \qquad \alpha_3 = u_3-\xi-ia_3. \tag{5.7}$$

The equation (5.6) determines the zeros of the dispersion equation.

Finding the solutions $\gamma_1, \gamma_2, \gamma_3, \gamma_4,$ of this equation by means of algebraic formulae an taking the contour integrals on the right-hand side of (4.15) we can write (4.16) in the form

$$E_k(t) = \sum_{l=0}^{4} \exp\{-i\gamma_l(t)\, t\} \operatorname{Re}_{z=\gamma_l} + \sum_{l=1}^{N} \exp-(i\eta_l(t)t) \operatorname{Re}_{z=\eta_l} \tag{5.8}$$

where η_l are the poles of ψ_1 and Ψ_1 which determine the individual behaviour as dependent on initial conditions $\bar{f}_k(v, 0)$ and $\bar{F}_k(v, 0)$ and symbol $\operatorname{Re}_{z=a}$ denotes the residue of the integral (4.15) at the point $z = a$.

VI. Discussion of the results

The numerical method of solving the Volterra integral equation (2.8) is based on difference method with the trapezium integration algorithm. The calculation has been performed for the Jackson model of distribution function (5.1) for electrons and ions, respectively. The self-consistent electric field as a function of time behaves differently for various times. We will discuss now the case when a plasma is stable at the time $t = 0$. In the first interval the field E_k is a decreasing function of time and its logarithmic decrement is also a decreasing function of time. In the second time interval, if the parameter a which is proportional to the temperature divided by the mass of a particle, is not to large, we have a growing field E_k.

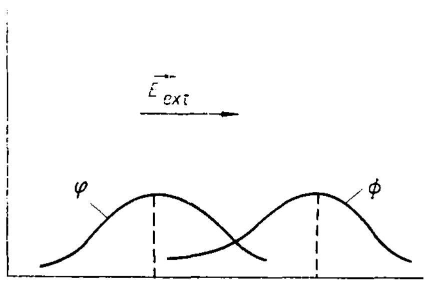

Fig. 1

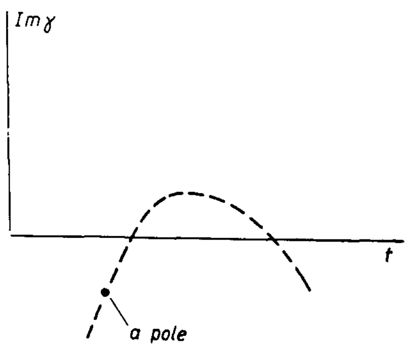

Fig. 2

In the third time interval the self-consistent electric field is again decreasing (see Fig. 7). The lengths of the respective time intervals depend on the initial conditions of a plasma (*i. e.* mean velocity, wave number and density), its material parameters (in the case of Jackson model and of equal masses of electrons and ions these parameters are: temperature, masses, electron plasma frequency) and the strength of the external field.

This behaviour is in agreement with the discussion performed by Fried, Gell-Mann, Jackson and Wyld [1].

One can get very easily the same qualitative results by analysing the dispersion relation (3.13). This gives us an exellent physical interpretation of above mentioned time inter-

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vals. At the beginning we have Landau damping with pole moving towards the real axis due to the separation of distribution functions, caused by the external field. At certain moment the pole reaches a real axis and we have then unstable behaviour corresponding to the second time interval. But as the separation increases (see Fig. 1) the interaction between the two streams of cherged particles become weaker and plasma oscillations are damped again, which happens in the third interval of time.

We note that, generally speaking, the thermal motion acts destructively on collective effects, including unstable oscillations. With increasing temperature the lengths of the second time interval decreases and even can vanish completely.

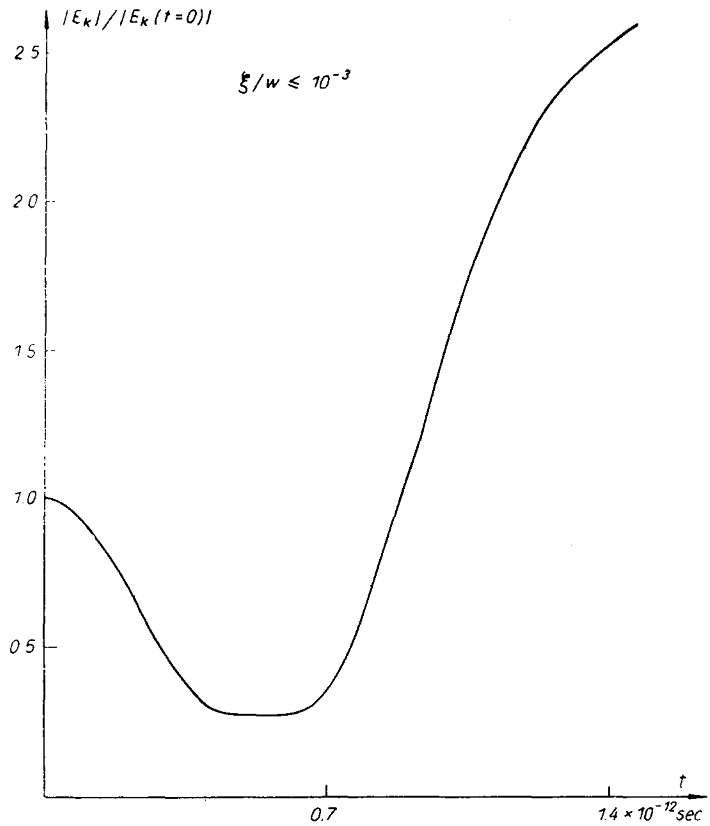

Fig. 3

The accuracy of the approximate solutions depends on the value of the dimensionless parameter $r = (\xi - w)/\Omega$, where $\xi = eEt/m$, $w = (u_e - u_i)/2$, $\Omega = \omega_p^2/k$, and u_e and u_i are mean velocities of electron and ions, respectively, at time $t = 0$ we have $u_e = u_1$ and $u_i = u_3$. In Fig. 3 we have a plot E_k *vs.* time for $\xi/w \leqslant 10^{-3}$ and for times in which the instability occurs, *i. e.* for times belonging to the second time interval. In this case the approximate solutions are in good agreement with the numerical solution (the relative error is of the order of $\xi/w \leqslant 10^{-3}$) and the corresponding plots overlap. Figure 4 represents

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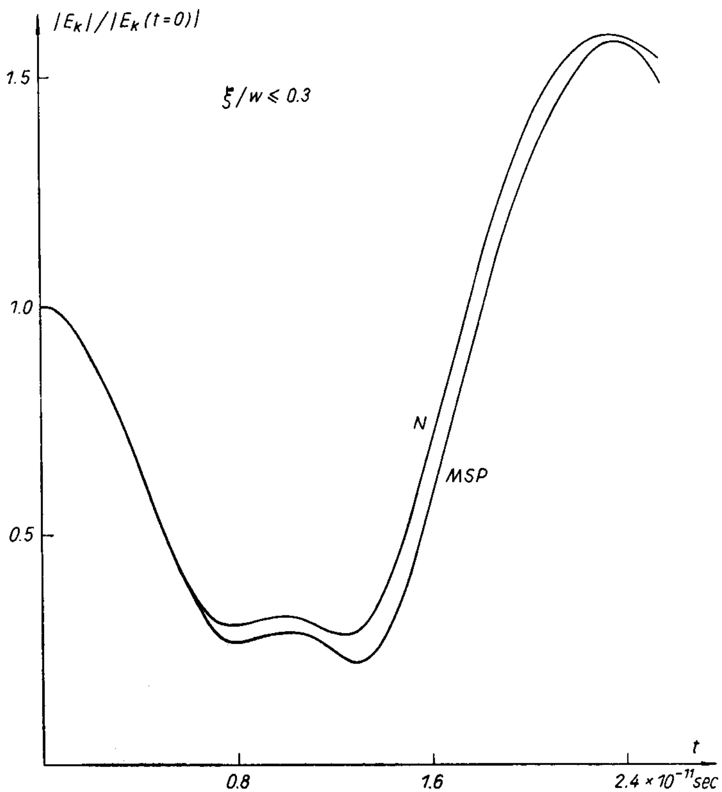
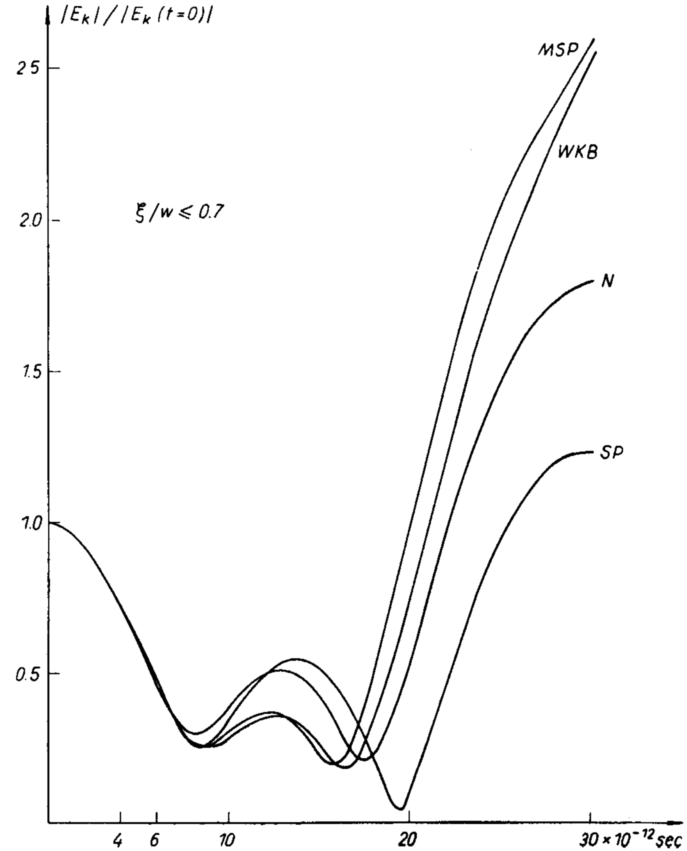


|E_k|/|E_k(t=0)|
1.5
1.0
0.5
ξ/w ≤ 0.3
N
MSP
t
0.8
1.6
2.4 × 10^-11 sec

Fig. 4



|E_k|/|E_k(t=0)|
2.5
2.0
1.5
1.0
0.5
ξ/w ≤ 0.7
MSP
WKB
N
SP
4
6
10
20
30 × 10^-12 sec

Fig. 5

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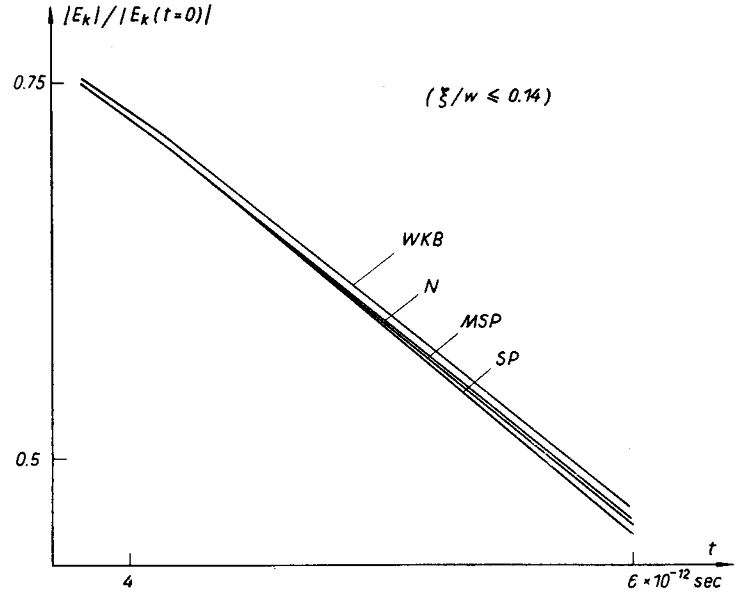

Fig. 6

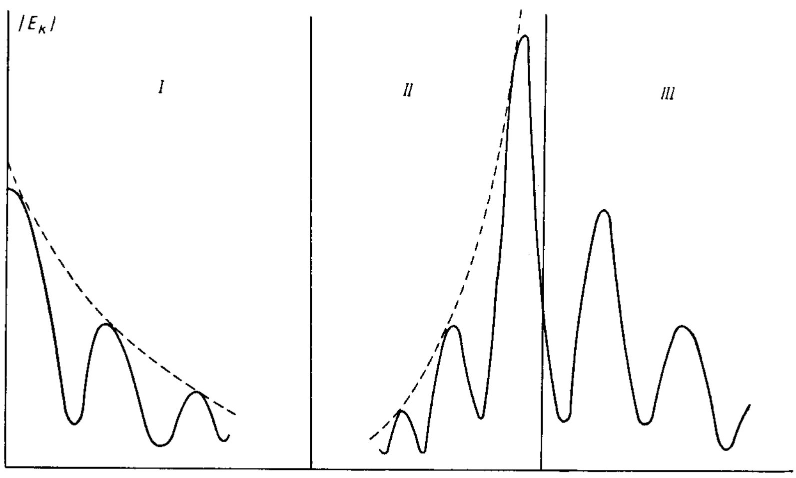

Fig. 7

the numerical solution compared with the approximate multiscale perturbation solution for $\xi/w \leqslant 0.3$, in which case the difference between the two corresponding plots in significant (upper curve correspond to the exact numerical solution). Figure 5 presents some other solutions of Eq. (2.8) for $\xi/w \leqslant 0.7$. The plots presented on this figure correspond to the multiscale perturbation (MSP) approximation, WKB approximation (phase integral method), exact numerical solution (N) and simple perturbation (SP) solution, respectively. Figure 6 present a fragment of Fig. 5 corresponding to the time interval from 3.5×10^{-12}

to 6×10^{-12} sec, and shows the behaviour of various approximate solutions for short times. As can be seen from Fig. 6 at the beginning ($t < 6\times10^{-12}$ sec or $\xi/w \leqslant 0.14$) the multiscale perturbation method is the best of the applied approximations. From Fig. 5 we can see that for times beyond the range of good accuracy ($\xi/w > 0.14$, or $t > 6\times10^{-12}$ sec) all approximations are quantitatively wrong, but qualitatively agree with each other and with the numerical solution. For simplicity we have put $m_e = m_i$ everywhere. In this case only one of the four roots of the dispersion equation can give rise to unstable bahaviour and this can occur only if

$$r^2 < 2 \tag{6.1}$$

and

$$1-\zeta^2-\sqrt{1-4\zeta^2} < r^2 < 1-\zeta^2+\sqrt{1-4\zeta^2} \tag{6.2}$$

where $\zeta = a/\Omega$. The length of the second interval of time is equal to $2\sqrt{1-4\zeta^2}$, but is zero if

$$\xi^2 > 1/4 \tag{6.3}$$

(For more detailed discussion of this problem see, for instance Ref. [8] Chap. 4).

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REFERENCES

[1] B. D. Fried, M. Gell-Mann, J. D. Jackson, H. W. Wyld., *J. Nuclear Energy*, (GB) Part C, Plasma Physics, **1**, 190 (1960).

[2] N. G. van Kampen, *Physica*, **21**, 949 (1955).

[3] K. M. Case, *Ann. Phys.* (USA), **1**, 293 (1959).

[4] R. Żelazny, *Ann. Phys.* (USA), **19**, 177 (1962).

[5] M. Trocheris, *Nuclear Fusion* (Internat.), **5**, 299 (1965).

[6] S. Bodner, E. Frieman, *Propagation of Instabilities in Plasmas*, paper in *The Seventh Lockheed Symposium on Magnetohydrodynamics*, E. Futterman ed. Stanford Univ. Press, Stanford 1963.

[7] J. D. Jackson, *J. Nucl. Energy*, Part C, Plasma Physics, **1**, 171 (1960).

[8] R. Balescu, *Statistical Mechanics of Charged Particles*, J. Willey and Sons N. York-London 1963.

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