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GREEN'S FUNCTIONS FOR THE BOSE SUPERFLUID AND THE RELATIONS BETWEEN KINETICAL COEFFICIENTS

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The Green's functions are obtained with the help of the hydrodynamic equations with viscous terms and from the formulae connecting average values of dynamical variables with retarded or advanced thermodynamical Green's functions. From the relations between the advanced and retarded Green's functions the relations for some kinetical coefficients are obtained.

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1. Introduction

In the paper [1] Bogoliubov proposed a method for the calculation of the Green's functions (named hereafter as *GF*-s) in the so-called hydrodynamic approximation. The *GF*-s were calculated for the system described by hydrodynamic equations without viscous terms.

The Bogoliubov formalism was used in Ref. [2]–[4] to calculate the *GF*-s for the normal and the superfluid systems described by hydrodynamic equations with viscous terms. The viscous terms are postulated as the most general forms which are linear in the sources of particles η, η^* and in the space derivatives of the density of particles ϱ, temperature θ, components of the velocity of the superfluid component v_s^α and normal component v_n^α. In the paper [3] some connections are found between the coefficients of these linear forms from the condition of the existence of thermodynamic equilibrium. In Ref. [5] it is pointed out that there is a possibility of finding some connections between the kinetical coefficients from the relations between the *GF*-s of the type, *e.g.* $\langle\langle\hat{\varrho}, \hat{\psi}\rangle\rangle$, $\langle\langle\hat{\psi}, \hat{\varrho}\rangle\rangle$ which can be calculated independently from the hydrodynamic equations. Such a connection was found for a special case in [6], namely for the coefficients of the "sources" η, η^*.

The aim of this paper is to obtain, by using the formalism proposed in [5], the Onsager-Khalatnikov relation between the coefficients of the second viscosity (this relation cannot be obtained from the consideration of the thermodynamic equilibrium used in [3] and incidentally the relation obtained in [6]. For this purpose we derive the hydrodynamic equations

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and calculate from them the suitable three-legs *GF*-s. Then from the connections between these *GF*-s we shall obtain relations between the dissipative coefficients for the Bose superfluid.

2. *Hydrodynamic equations*

Consider a superfluid Bose-system with the Hamiltonian

$$\hat{H}[\delta\eta,\, \delta U,\, \delta \boldsymbol{A}] = \hat{H} + \delta\hat{H}_t^1[\delta\eta(r,t);\, \delta U(r,t);\, \delta\boldsymbol{A}(r,t)] \tag{1}$$

$$\hat{H} = \frac{1}{2m}\int \nabla\psi^{+}(r,t)\,\nabla\psi(r,t)\,dr + \frac{1}{2}\int \Phi(|r-r'|)\,\psi^{+}(r',t)\,\psi^{+}(r',t)\,\psi(r',t)\,\psi(r,t)drdr' -$$

$$-\lambda\int \hat{\varrho}(t,r)\,dr \quad \delta H_t^1 = \int \delta U(r,t)\,\hat{\varrho}(r,t)\,dr - \frac{1}{m}\int \hat{\boldsymbol{j}}(t,r)\delta\boldsymbol{A}(t,r)dr -$$

$$-\frac{1}{2m}\int \psi^{+}(t,r)\,\delta A^2\,(t,r)\,\psi(t,r)dr +$$

$$+\int \{\delta\eta(t,r)\psi^{+}(t,r) + \delta\eta^{*}(t,r)\psi(t,r)\}\,dr$$

$$\hat{\boldsymbol{j}}(t,r) = \frac{i}{2}\,[(\nabla\psi^{+}(t,r)\,\psi(t,r) - \psi^{+}(r,t)\,(\nabla\psi(r,t))] - \hat{\varrho}(r,t)\,\delta\boldsymbol{A}(r,t) \equiv \hat{\boldsymbol{j}}^{\circ}(r,t) -$$

$$-\hat{\varrho}(r,t)\delta\boldsymbol{A}(r,t) \qquad \hat{\varrho}(r,t) = \psi^{+}(r,t)\psi(r,t)$$

In the Hamiltonian (1) we have an additional term with "sources of particles" $\delta\eta$, $\delta\eta^*$, with an external scalar potential δU and an external vector potential $\delta\boldsymbol{A}$. We take all these functions as given. The adiabatic introduction of the infinitesimal term $\delta\hat{H}_t^1$ gives the small deviation of the system from the thermodynamic equilibrium. (We write r istead $\boldsymbol{r}$ and $\hbar = 1$).

From the Hamiltonian (1) we can find the equations of motion for $\psi(r,t)$ and $\psi^{+}(r,t)$, *e.g.*:

$$i\,\frac{\partial\psi(r,t)}{\partial t} = -\frac{\Delta\psi(r,t)}{2m} + \int \Phi(|r-r'|)\,\psi^{+}(t,r')\,\psi(t,r')\,dr'\psi(t,r) +$$

$$+[\delta U(t,r) - \lambda]\,\psi(r,t) + \frac{i}{m}\,(\nabla\psi(r,t))\,\delta\boldsymbol{A}(r,t) + \frac{i}{2m}\,\psi(r,t)\,\nabla\delta\boldsymbol{A}(r,t) +$$

$$+\frac{1}{2m}\,\psi(r,t)\,\delta A^2(r,t) + \eta(r,t) \tag{2}$$

With the aid of Eqs. (2) we can obtain the hydrodynamic equations with viscous terms for Bose superfluid [2]. For the case of infinitesimal δH_t^1 we can consider only the linearized

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hydrodynamic equations for the variations $\delta\varrho$, $\delta\theta$, $\delta_n v$, δv_s describing the deviation from the thermodynamic equilibrium. The linearized equations have the form [2], [7]

$$\frac{\partial\delta\varrho\,(t,r)}{\partial t}+\varrho_s\operatorname{div}\delta v_s(t,r)+\varrho_n\operatorname{div}\delta v_n(t,r)=i\sqrt{\varrho_0}\,[\delta\eta^*(t,r)-\delta\eta(t,r)]$$

$$m\varrho_s\frac{\partial\delta v_s^\alpha(t,r)}{\partial t}+m\varrho_n\frac{\varrho\partial\delta v_n^\alpha(r,t)}{\partial t}=-\left(\frac{\partial P}{\partial\theta}\right)_\varrho\frac{(\partial\delta\theta(r,t)}{\partial r_\alpha}\left(\frac{\partial p}{\partial\varrho}\right)_\theta\frac{\partial\delta\varrho(t,r)}{\partial r_\alpha}+$$

$$+\zeta_1\varrho_s\frac{\partial}{\partial r_\alpha}\operatorname{div}\delta v_s+\eta\Delta\delta_n^\alpha+\left(\frac{1}{3}\eta-\zeta_1\varrho_s+\zeta_2\right)\frac{\partial}{\partial r_2}\operatorname{div}v_n+$$

$$+\sqrt{\varrho_0}\,[\varrho B^*-i(\zeta_1-\varrho\zeta_3)]\frac{\partial\delta\eta^*(r,t)}{\partial r_\alpha}+\sqrt{\varrho_0}\,[\varrho B+i(\zeta_1-\varrho\zeta_3)]\frac{\partial\delta\eta\,(t,r)}{\partial r_\alpha}-$$

$$-\varrho\left(\frac{\partial\delta A^\alpha(t,r)}{\partial t}+\frac{\partial\delta U(r,t)}{\partial r_\alpha}\right)$$

$$\varrho\frac{\partial\delta S(r,t)}{\partial t}+S\frac{\partial\delta\varrho(t,r)}{\partial t}+\varrho\,S\operatorname{div}\delta v_n=\frac{\varkappa}{\theta}\Delta\delta\theta(r,t)$$

$$m\varrho\frac{\partial\delta v_s^\alpha(r,t)}{\partial t}=\left[\varrho S-\left(\frac{\partial P}{\partial\theta}\right)_\varrho\right]\frac{\partial\delta\theta(r,t)}{\partial r_\alpha}-\left(\frac{\partial P}{\partial\varrho}\right)_\theta\frac{\partial\delta\varrho(t,r)}{\partial r_\alpha}+$$

$$+\zeta_3\varrho_s\varrho\frac{\partial}{\partial r_\alpha}\operatorname{div}dv_s+\varrho_s(\zeta_4-\varrho_s\zeta_3)\frac{\partial}{\partial r_\alpha}\operatorname{div}\delta v_n+$$

$$+\sqrt{\varrho_0}B\varrho\frac{\partial\eta(t,r)}{\partial r_\alpha}+\sqrt{\varrho_0}B^*\varrho\frac{\partial\eta^*(r,t)}{\partial r_\alpha}-\varrho\left(\left(\frac{\partial\delta A^\alpha(t,r)}{\partial t}\right)+\frac{\partial\delta U(t,r)}{\partial r_\alpha}\right)\qquad(3)$$

where $\varrho=\varrho_s+\varrho_n$, $\delta j=m\varrho_s\delta v_s+m\varrho_n\delta v_n$ and the indices s, n correspond to the superfluid and normal components, resp., ϱ_0 is the density of the condensate in the thermodynamic equilibrium, $v_s=\frac{1}{m}(\nabla\chi-\delta A)$ (where we put $\langle\psi\rangle=ae^{i\chi}$). The coefficient of thermal conductivity $\varkappa$ and the viscosity coefficients η, ζ_1, ζ_2, ζ_3, ζ_4, B, B^* are functions of ϱ° and θ° (in the Eqs (3) we omit the upper index).

In order to introduce δH_τ^1 adiabatically we take the variations $\delta\eta(t,r)$, $\delta U(t,r)$, $\delta A^\alpha(t,r)$ in the form

$$\delta f(t,r)=e^{-i\omega t+\varepsilon t+ikr}\delta f_k+e^{i\omega t+\varepsilon t-ikr}\delta f_{-k}\qquad(4)$$

where $\varepsilon>0$, $\varepsilon\to 0$.

If we substitute (4) into (3) and write also $\delta\varrho$, $\lambda\theta$, δV_s^α, δV_n^α in the form (4) we obtain the following set of four equations for four unknowns

$$-\omega x_1+\varrho_s x_2+\varrho_n x_3=\sqrt{\varrho_0}(\eta_{-k}^*-\eta_k)$$

$$-k^2\left(\frac{\partial P}{\partial\varrho}\right)_\theta x_1+(\omega m\varrho_s-i\zeta_1\varrho_s k^2)x_2+\left[\omega m\varrho_n-i\left(\frac{4}{3}\eta-\varrho_s\zeta_1+\zeta_2\right)k^2\right]x_3-$$

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$$-k^2\left(\frac{\partial P}{\partial \theta}\right)_{\varrho} x_4 = \sqrt{\varrho_0}\, k^2 \varrho\, (B^* \eta^*_{-k} + B\eta_k) - i\sqrt{\varrho_0}\, k^2 (\zeta_1 - \varrho\zeta_3)(\eta^*_{-k} - \eta_k) + \varrho(k^2 \delta U_k - \omega \boldsymbol{k} \delta \boldsymbol{A}_k)$$

$$-k^2\left(\frac{\partial P}{\partial \varrho}\right)_{\theta} x_1 + (\omega m\varrho - i\varrho\varrho_s\,\zeta_3 k^2)x_2 - ik^2\varrho\,(\zeta_4 - \zeta_3\varrho_s)x_3 -$$

$$-k^2\left[\left(\frac{\partial P}{\partial \theta}\right)_{\varrho} - \varrho S\right] x_4 = \sqrt{\varrho_0}\, k^2\varrho\,(B^*\eta^*_{-k} + B\eta_k) + \varrho(k^2\,\delta U_k - \omega \boldsymbol{k}\delta\boldsymbol{A}_k)$$

$$\omega\left[S + \varrho\left(\frac{\partial S}{\partial \varrho}\right)_{\theta}\right] x_1 - \varrho S x_3 + \left[\omega\varrho\left(\frac{\partial S}{\partial \theta}\right)_{\varrho} - \frac{i\varkappa}{\theta}k^2\right] x_4 = 0 \tag{5}$$

where x_i are $\delta\varrho_k$, $\boldsymbol{k}\delta\boldsymbol{v}_s(k)$, $\boldsymbol{k}\delta\boldsymbol{v}_n(k)$, $\delta\theta_k$, respectively. As we need the hydrodynamic equations for negative times we have changed the signs of the kinetical coefficients in (3).

We see that the solutions of (5) are proportional to $\delta\eta^*_{-k}$, $\delta\eta_k$, δU_k, $\delta\boldsymbol{A}_k$. It is convenient to write the solutions for $\delta\varrho_k$, $\boldsymbol{k}\delta\boldsymbol{v}_s(k)$, $\boldsymbol{k}\delta\boldsymbol{v}_n(k)$ in the form

$$\delta\varrho_k = R^\eta\delta\eta_k + R^{\eta*}\delta\eta^*_{-k} + R^U\delta U_k + \boldsymbol{R}^A\delta\boldsymbol{A}_k$$
$$\boldsymbol{k}\delta\boldsymbol{v}_s(k) = V^\eta_s\delta\eta_k + V^{\eta*}_s\delta\eta^*_{-k} + V^U_s\delta U_k + \boldsymbol{V}^A_s\delta\boldsymbol{A}_k$$
$$\boldsymbol{k}\delta\boldsymbol{v}_n(k) = V^\eta_n\delta\eta_k + V^{\eta*}_n\delta\eta^*_{-k} + V^U_n\delta U_k + \boldsymbol{V}^A_n\delta\boldsymbol{A}_k \tag{6}$$

where the interesting derivatives with respect to $\delta\eta_k$ $\delta\eta^*_{-k}$ δU_k are equal to

$$R^\eta = \frac{\delta\varrho_k}{\delta\eta_k} = \frac{\sqrt{\varrho_0}}{\Omega}(-W_1 + BW_2)$$

$$R^{\eta*} = \frac{\delta\varrho_k}{\delta\eta^*_{-k}} = \frac{\sqrt{\varrho_0}}{\Omega}(W_1 + B^*\,W_2) \tag{7}$$

where

$$W_1 = \omega(c_2^2k^2 - \omega^2) + ik^2\,\omega^2\left[\frac{1}{m\varrho_n}\left(\frac{4}{3}\,\eta - \varrho\zeta_1 + \zeta_2 + \varrho^2\zeta_3 - \varrho_s\zeta_4\right) + \frac{\varkappa}{c_V\varrho}\right] - i\zeta_3\,k^4c_2^2\,\frac{\varrho}{m}$$

$$W_2 = \frac{1}{m}\,k^2\varrho\,(\omega^2 - c_2^2\,k^2)$$

$$V^U_s = \boldsymbol{k}\,\frac{\delta\boldsymbol{v}_s(k)}{\delta U_k} = \frac{\omega k^2}{m\Omega}\left\{\omega^2 - c_2^2k^2 - i\omega k^2\left[\frac{1}{m\varrho_n}\left(\frac{4}{3}\,\eta - \varrho_s\zeta_1 + \zeta_2 + \varrho\varrho_s\zeta_3 - \varrho\zeta_4\right) + \frac{\varkappa}{\varrho C_V}\right]\right\} \tag{8}$$

and

$$V^{A\alpha}_s = \boldsymbol{k}\,\frac{\delta\boldsymbol{v}_s(k)}{\delta A^\alpha_k} = -\frac{\omega k^\alpha \boldsymbol{k}}{k^2}\,\frac{\delta\boldsymbol{v}_s(k)}{\delta U_k} \tag{9}$$

Instead of the formulas for $\frac{\delta \boldsymbol{v}_{n,s}}{\delta \eta_k}$, $\frac{\delta \boldsymbol{v}_{n,s}}{\delta \eta^*_{-k}}$ we can write at once

$$\boldsymbol{k}\,\frac{\delta \boldsymbol{j}_k}{\delta \eta^*_{-k}} = m\varrho_s \boldsymbol{k}\,\frac{\delta \boldsymbol{v}_s(k)}{\delta \eta^*_{-k}} + m\varrho_n \boldsymbol{k}\,\frac{\delta \boldsymbol{v}_n(k)}{\delta \eta^*_{-k}}$$

$$= m\varrho_s V_s^{\eta^*} + m\varrho_n V_n^{\eta} = \frac{m\sqrt{\varrho_0}k^2}{\Omega}\,(W_3 + B^* W_4) \tag{10}$$

$$\boldsymbol{k}\,\frac{\delta \boldsymbol{j}_k}{\delta \eta_k} = \frac{m\sqrt{\varrho_0}\,k^2}{\Omega}\,(-W_3 + BW_4) \tag{11}$$

where

$$W_3 = -(\omega^2 - c_2^2 k^2)\,c_1^2 - \frac{i\omega^3}{m}\,(\zeta_1 - \varrho\zeta_3) +$$

$$+ ik^2\omega\left[\frac{c_1^2\varrho_s}{m\varrho\varrho_n}\left(\frac{4}{3}\,\eta - \varrho\,\zeta_1 + \zeta_2 + \varrho^2\zeta_3 - \varrho\zeta_4\right) + c_1^2\,\frac{\varkappa}{c_V\varrho} + \frac{c_2^2}{m\varrho}\left(\frac{4}{3}\,\eta + \zeta_2 - \varrho^2\zeta_3\right)\right]$$

$$W_4 = \frac{\omega\varrho^2}{m}\,(\omega^2 - c_2^2 k^2)$$

$c_1^2 = \frac{1}{m}\left(\frac{\partial P}{\partial \varrho}\right)_s$ and $c_2^2 = \frac{S^2\varrho_s\theta}{m\varrho_n c_V}$ are the approximate formulas for the velocities of the first and second sound, respectively, and

$$\Omega = \omega^4 - ik^2\omega^3\left[\frac{1}{m\varrho_n}\left(\frac{4}{3}\,\eta - \varrho_s\zeta_1 + \zeta_2 + \varrho\varrho_s\zeta_3 - \varrho_s\zeta_4\right) + \frac{\varkappa}{c_V\varrho}\right] -$$

$$-k^2\omega^2(c_1^2 + c_2^2) + ik^4\omega\left[c_1^2\,\frac{\varrho_s}{m\varrho\varrho_n}\left(\frac{4}{3}\,\eta + \zeta_2 - \varrho\zeta_1 + \varrho^2\zeta_3 - \varrho\zeta_4\right) + \right.$$

$$\left. + c_1^2\,\frac{\varkappa}{\varrho c_V} + \frac{c_2^2}{m\varrho}\left(\frac{4}{3}\,\eta + \zeta_2\right)\right] + k^4 c_1^2 c_2^2 \tag{12}$$

The remaining derivatives with respect to δU_k, δA_k^α will be considered in [8].

3. *Green's functions*

If the thermodynamic equilibrium of the system is violated by an adiabatic introduced disturbance $\delta \hat{H}^1_\tau[\delta\eta, \delta U, \delta A]$, the following equation connects the variation of the mean values $\delta\langle \hat{b}(t, r)\rangle$ with the *GF*-s (see *e.g.* Ref. [9], [3])

$$\delta\langle \hat{b}(t)\rangle = \int_{-\infty}^{\infty} \langle\!\langle \hat{b}(t);\, \delta\hat{H}^1_\tau(\tau)\rangle\!\rangle^r d\tau$$

where the operators $\hat{b}(t)$ and $\delta\hat{H}^1_\tau(\tau)$ are in the Heisenberg representation with the Hamiltonian $\hat{H}$. Using (1), (4) we obtain the equality

$$\delta\langle b(k,\omega)\rangle \equiv \delta b_k = 2\pi \langle\langle \hat{b}_k; \hat{\varrho}_{-k}\rangle\rangle^r_\omega \delta U_k - \frac{2\pi}{m}\langle\langle \hat{b}_k; \hat{\boldsymbol{j}}^0_{-k}\rangle\rangle^r_\omega \delta \boldsymbol{A}_k + $$

$$+ 2\pi \langle\langle \hat{b}_k; \hat{\boldsymbol{a}}^+_k\rangle\rangle^r_\omega \delta\eta_k + 2\pi \langle\langle \hat{b}_k; \hat{\boldsymbol{a}}_{-k}\rangle\rangle^r_\omega \delta\eta^*_{-k} \tag{13}$$

in the Fourier components, where

$$\hat{\varrho}_k = \frac{1}{\sqrt{V}}\sum_q a^+_q a_{q+k} \qquad \hat{\boldsymbol{j}}^0_k = \frac{1}{\sqrt{V}}\sum_p \boldsymbol{p} a^+_{q-\frac{k}{2}} a_{q+\frac{k}{2}}$$

For $\hat{\varrho}(t,r)$ and $\hat{\boldsymbol{j}}(t,r)$ we have from (13)

$$\frac{\delta\varrho_k}{\delta\eta_k} = 2\pi \langle\langle \hat{\varrho}_k; \hat{\boldsymbol{a}}^+_k\rangle\rangle^r_\omega \qquad \frac{\delta\boldsymbol{j}_k}{\delta\eta_k} = 2\pi \langle\langle \hat{\boldsymbol{j}}^0_k; \hat{\boldsymbol{a}}^+_k\rangle\rangle^r_\omega$$

$$\frac{\delta\varrho_k}{\delta\eta^*_{-k}} = 2\pi \langle\langle \hat{\varrho}_k; \hat{\boldsymbol{a}}_{-k}\rangle\rangle^r_\omega \qquad \frac{\delta\boldsymbol{j}_k}{\delta\eta^*_{-k}} = 2\pi \langle\langle \hat{\boldsymbol{j}}^0_k; \hat{\boldsymbol{a}}_{-k}\rangle\rangle^r_\omega \tag{14}$$

On the other hand these three-legs *GF*-s can be found if we calculate them from the hydrodynamic equation for the velocity of the superfluid component which can be written in the form [1]

$$\delta\boldsymbol{v}_s(t,\boldsymbol{r}) - \frac{1}{m}\delta\boldsymbol{A}(t,\boldsymbol{r}) = \frac{i}{2m\sqrt{\varrho_0}}\left(\frac{\partial\delta\langle\psi^+(t,\boldsymbol{r})\rangle}{\partial\boldsymbol{r}} - \frac{\partial\delta\langle\psi(t,\boldsymbol{r})\rangle}{\partial\boldsymbol{r}}\right) \tag{15}$$

If we put in (13) $\hat{b}$ equal to $\hat{\psi}^+$ and ψ we will obtain from (15)

$$\frac{\delta v^\alpha_s(k)}{\delta U_k} = \frac{\pi k^\alpha}{m\sqrt{\varrho_0}}\langle\langle \hat{a}_k - \hat{a}^+_{-k}; \hat{\varrho}_{-k}\rangle\rangle^r_\omega$$

$$\frac{\delta v^\alpha_s(k)}{\delta A^\beta_k} + \frac{1}{m}\delta_{\alpha\beta} = -\frac{\pi k^\alpha}{m^2\sqrt{\varrho_0}}\langle\langle \hat{a}_k - a^+_{-k}; \hat{j}^{0\beta}_{-k}\rangle\rangle^r_\omega \tag{16}$$

The comparison of (14), (16) with (7), (8), (9) leads to the formulae for *GF*-s (in the hydrodynamical approximation).

Now we want to find the relations for *GF*-s of the type $\langle\langle\varrho, a\rangle\rangle^r_\omega$ and $\langle\langle a, \varrho\rangle\rangle^r_\omega$. For this purpose we use the relation (see *e.g.* Ref. [10])

$$\langle\langle b_k, d_{-k}\rangle\rangle_E = \langle\langle d_{-k}, b_k\rangle\rangle_{-E} \tag{17}$$

(E is complex and equal to $E = \omega + i\varepsilon$) and then we calculate the suitable advanced *GF*-s. The equation (3), (4) describe the system, which is in the thermodynamic equilibrium for $t = -\infty$ and are written for negative times. For the calculation of the advanced *GF*-s we must find $\delta\varrho_k$, $\delta\boldsymbol{v}_s(k)$ which vanish for $t = +\infty$. They are expressed by advanced *GF*-s and must be found from the hydrodynamic equations for positive times. This means that we must change the signs of the kinetical coefficients in (5).

We are interested in the advanced GF-s $\langle\langle d_{-k}, b_k\rangle\rangle^a_{-\omega}$. We see that if we change $\omega\to-\omega$, $k\to -k$ in (5) (after the change of signs of kinetical coefficients) we obtain the equations for $\delta\varrho^{\text{adv}}(-\omega,-k)$, *etc.*, which are similar to (5) for $\delta\varrho^{\text{ret}}_{\varrho}(\omega, k)$, *etc.* (only the signs before $\delta\eta$, $\delta\eta^*$ are changed). For the four-legs GF-s these arguments lead to some identities [8]. In our case we have

$$\boldsymbol{k}\,\frac{\delta \boldsymbol{v}_s(k)}{\delta U_k} = \frac{\pi k^2}{m\sqrt{\varrho_0}}\langle\langle \hat{a}_k-\hat{a}^+_{-k};\,\hat{\varrho}_{-k}\rangle\rangle^r_\omega$$

$$= \frac{k^2\pi}{m\sqrt{\varrho_0}}\left(\langle\langle\varrho_{-k}, a_k\rangle\rangle^a_{-\omega}-\langle\langle\varrho_{-k}, a^+_{-k}\rangle\rangle^a_{-\omega}\right)$$

$$=\frac{k^2}{2m\sqrt{\varrho_0}}\left(\frac{\delta\varrho^{\text{adv}}(-k,-\omega)}{\delta\eta^*(k,-\omega)}-\frac{\delta\varrho^{\text{adv}}(-k,-\omega)}{\delta\varrho(-k,-\omega)}\right)=\frac{k^2}{2m\sqrt{\varrho_0}}\left(\frac{\partial\varrho^{\text{ret}}\,k,\omega)}{-\delta\eta^*(-k,+\omega)}-\right.$$

$$\left.-\frac{\delta\varrho^{\text{ret}}(k,\omega)}{-\delta\eta(k,\omega)}\right)\equiv\frac{k^2}{2m\sqrt{\varrho_0}}\left(\frac{\delta^{\text{ret}}\varrho_k}{\delta\eta_k}-\frac{\delta^{\text{ret}}\varrho_k}{\delta\eta^*_{-k}}\right). \tag{18}$$

Now if we put the solutions (7), (8) into (18) we find the expression

$$\omega^3\left(i\zeta_1-i\zeta_4-i\varrho\zeta_3+\frac{B-B^*}{2}\varrho\right)+k^2\varrho c_2^2\left(i\zeta_3-\frac{B-B^*}{2}\right)=0$$

Because ω and k are independent, we have for the imaginary part of B (see also [6])

$$\operatorname{Im} B=\zeta_3 \tag{19}$$

and the Onsager–Khalatnikov relation

$$\zeta_1=\zeta_4. \tag{20}$$

From (16), (17) and (5) we find analogously to (18) that

$$k^2+m\sum_{\alpha,\beta}k^\alpha k^\beta\,\frac{\delta v^\alpha_s(k)}{\delta A^\beta_k}=\frac{\pi k^2}{2\sqrt{\varrho_0}}\left(\boldsymbol{k}\,\frac{\delta \boldsymbol{j}_k}{\delta\eta^*_{-k}}-\boldsymbol{k}\,\frac{\delta\boldsymbol{j}_k}{\delta\eta_k}\right)$$

After inserting (9), (10), (11) into this equation we obtain results identical to (19) and (20).

The relation (20) was obtained for the first time by Khalatnikov [11] and is the Onsager symmetry relation.

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