COHERENCE OF ELECTROMAGNETIC FIELD AND THE RADIATION SOURCE STRUCTURE

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The coherence effects of optical fields have so far been related to the finite spectral width of radiation and to the source size. These two effects are called temporal coherence and spatial coherence effects, respectively. On the basis of earlier investigations on wide-angle interference we can however expect another effect related to the multipolarity of the source. This effect of multipole coherence, which plays a role both in vector and scalar formulation of the theory, is discussed in connexion with the effects mentioned above. The electromagnetic degree of coherence is evaluated for a simple model of the source. In the concluding part of this paper a generalization of the van Cittert-Zernike theorem for multipole radiation is derived. It is indicated that for the fields of narrow angular spread a scalar theory is applicable.

1. *Introduction*

The modern approach to coherence theory is based on investigations of electromagnetic field fluctuations. Optical coherence phenomena may be said to be manifestations of these fluctuations [1].

In the statistical description of an electromagnetic field optical disturbances at a fixed point of space are treated as random functions of time. Correlation functions of any given order correspond to appropriate statistical moments. In the case of fields generated by thermal sources the complete statistical description is contained in the second-order correlation function. The reason for this is that the thermal radiation is characterized by a normal (Gaussian) probability distribution for which all higher-order moments are determined by the moments of the first two orders.

With the development of lasers and new light detectors other types of radiation, of a different statistical nature, have been studied. In the following we shall be concerned, however, only with the second-order coherence theory.

The mutual coherence function, a basic quantity in the scalar theory of coherence, is defined as

$$\Gamma(P_1, P_2, \tau) = \langle (P_1, t+\tau) V^*(P_2, t)\rangle^{1},$$

where $V(P_1, t)$ and $V(P_2, t)$ are the field values (in complex representation) at space-points

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[1] In vector formulation of the theory electromagnetic field correlation tensors are defined.

P_1 and P_2, and the sharp brackets denote the averaging operation over an ensemble of all possible realizations of the field. For stationary fields $\Gamma(P_1, P_2, \tau)$ must be independent of the origin of time, in this case we assume as a rule an ergodic type hypothesis and replace the ensemble average by the infinite time average.

Since the mutual coherence function obeys the wave equation, we may say that the correlations between the field disturbances, as well as the disturbance themselves, are propagated in the form of waves. This enables us to investigate partially coherent fields without entering into the underlying mechanism of the origin of coherence within the source itself. To solve any problem of determining $\Gamma(P_1, P_2, \tau)$ it is only necessary to specify it on a boundary surface. Hitherto this type of investigation has been prevailing in the development of the coherence theory and little attention has been paid to a possible connexion between the coherence properties of radiation field and the properties of the source. It is the purpose of this paper to fill this gap somewhat.

The coherence effects of optical fields have so far been related to the finite spectral width of radiation and to the source size [1], [2]. These two effects are called temporal coherence and spatial coherence effects, respectively. Although a complete separation of these effects is, in general, not possible[2], a partial separation proves useful in many cases. The temporal coherence is represented by $\Gamma(P_1, P, \tau)$ and can be measured by the Michelson two-beam interferometer. The measurement of $\Gamma(P_1, P_2, 0)$, by a Young type experiment for example, will yield information about the spatial coherence. The appearence of these two effects can be explained by by the following argument: Radiation of finite spectral width cannot be coherent since its various Fourier components are not correlated. Similarly, an extended source cannot radiate coherently due to the lack of correlation between its various spatial elements [4].

It follows from these considerations that strictly monochromatic radiation emanating from a point source should always be completely coherent, *cf.* [2] and [5]. We know, however, from the earlier investigation on wide angle interference (see [6]–[8] for the discussion of the theory and [9]–[12] for the description of experiments) that the visibility of interference fringes depends on the multipolarity of the source and the angle formed by the light beams, *cf.* also [13]. Since the visibility of fringes is directly related to the degree of coherence, we can expect that the multipole radiation will be partially coherent even if it is strictly monochromatic. This new effect, which should appear not only for electromagnetic multipole radiation but also for scalar multipole radiation, will be termed an effect of multipole coherence [14].

In general, we may study the multipole radiation emanating from an extended quasi-monochromatic source. All three effects must then be taken into account. We shall base our approach on the concept of an electromagnetic degree of coherence, introduced by Karczewski in 1963 [15]. This is a measurable scalar quantity which determines the degree of correlation between the electromagnetic disturbances at two space points. In the case when angular dimensions of the source cannot be neglected we shall derive a generalization of the van Cittert-Zernike theorem [16] for multipole radiation.

[2] It appears, for example, that this separation is not invariant with respect to the Lorentz transformation [3].

2. Temporal and multipole coherence of electromagnetic feld

Consider the distribution of charge $\varrho(\boldsymbol{r}, t)$, current $\boldsymbol{J}(\boldsymbol{r}, t)$ and magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$ confined within a bounded spatial region[3]. This region will constitute the source of radiation. The radiation field from such a collection of charge and current can be expressed as a superposition of 2^l multipole fields, the 2^l pole field being characterized by $2l+1$ complex amplitudes a_{lm}. In quantum theory this corresponds to the classification of emitted radiation according to the total angular momentum value, l, and the z-component of angular momentum, m [17]. For a given value of l two types of fields of different parity occur: the field of parity $(-1)^l$, termed electric multipole field, and the field of parity $(-1)^{l+1}$, termed magnetic multipole field.

We shall assume that the radiation is unpolarized and has an isotropic intensity distribution. Our aim is to calculate the degree of coherence for quasi-monochromatic radiation from the 2^l pole in the far zone.

We choose a spherical coordinate system (r, θ, φ) with the center somewhere within the source. We assume, to begin with, that the distance between the observation point and the source is so large that the angular dimensions of the source can be neglected. For a simple harmonic electromagnetic field, of frequency $\nu = kc/2$, the electric field strength is given by the following expansion [18]

$$\boldsymbol{E}_l^{(E)}(\boldsymbol{r}, \nu) = \frac{e^{ikr}}{r} \sum_{m=-l}^{l} (-1)^{l+1} a_{lm}^{(E)}(\nu) \boldsymbol{X}_{lm}(\theta, \varphi) \times \boldsymbol{n} \tag{1.a}$$

if the radiation is due to an electric 2^l pole, and

$$\boldsymbol{E}_l^{(M)}(\boldsymbol{r}, \nu) = \frac{e^{ikr}}{r} \sum_{m=-l}^{l} (-1)^{l+1} a_{lm}^{(M)}(\nu) \boldsymbol{X}_{lm}(\theta, \varphi) \tag{1.b}$$

if the radiation is due to a magnetic 2^l pole.

In the above formulas $\boldsymbol{n} = \boldsymbol{r}/r$, and $\boldsymbol{X}_{lm}$ denote the spherical vector harmonics, given in terms of the scalar spherical harmonics Y_{lm} by

$$\boldsymbol{X}_{lm}(\theta, \varphi) = \frac{(-i)}{\sqrt{l(l+1)}} \boldsymbol{r} \times \operatorname{grad} Y_{lm}(\theta, \varphi). \tag{2}$$

The amplitudes $a_{lm}^{(E)}$ and $a_{lm}^{(M)}$, depending on the current and magnetization distributions, are as follows

$$a_{lm}^{(E)}(\nu) = \frac{4\pi ik}{c} \int j_l(kr) \boldsymbol{X}_{lm}^*(\theta, \varphi) [\operatorname{rot} J(\boldsymbol{r}, \nu) + c \operatorname{rot} \operatorname{rot} M(\boldsymbol{r}, \nu)] \, d\boldsymbol{r}, \tag{3.a}$$

$$a_{lm}^{(M)}(\nu) = -4\pi k^2 \int j_l(kr) \boldsymbol{X}_{lu}^*(\theta, \varphi) \left[\operatorname{rot} \boldsymbol{M}(\boldsymbol{r}, \nu) + \frac{1}{ck^2} \operatorname{rot} \operatorname{rot} J(\boldsymbol{r}, \nu) \right] d\boldsymbol{r} \tag{3.b}$$

[3] The charge and current densities are related by the continuity equation. Magnetization is related to the magnetization current by $\boldsymbol{J}_M = c \operatorname{rot} \boldsymbol{M}$.

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where $j_l(kr)$ are the spherical Bessel functions. Quasi-monochromatic fields of mean frequency $\nu_0 = k_0 c/2\pi$ and spectral width $\Delta\nu$ $(\Delta\nu/\nu_0 \ll 1)$ will be represented as analytic sygnals [16]. By virtue of (1.a) and (1.b) we get for the radiation of electric and magnetic 2^l poles, respectively,

$$\boldsymbol{E}_l^{(E)}(\boldsymbol{r}, t) = \frac{(-1)^{l+1}}{k_0 r} \sum_{m=-l}^{l} a_{lm}^{(E)}\left(t - \frac{r}{c}\right) \boldsymbol{X}_{lm} \times \boldsymbol{n} \tag{4.a}$$

$$\boldsymbol{E}_l^{(M)}(\boldsymbol{r}, t = \frac{(-1)^{l+1}}{k_0 r} \sum_{m=-l}^{l} a_{lm}^{(M)}\left(t - \frac{r}{c}\right) \boldsymbol{X}_{lm} \tag{4.b}$$

where

$$a_{lm}^{(E)}(t) = \int_0^\infty a_{lm}^{(E)}(\nu) e^{-2\pi i\nu t} d\nu, \tag{5.a}$$

$$a_{lm}^{(M)}(t) = \int_0^\infty a_{lm}^{(M)}(\nu) e^{-2\pi i\nu t} d\nu. \tag{5.b}$$

The complex degree of coherence of the electromagnetic field is defined by Karczewski as follows

$$\delta(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{\Delta(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)}{[\Delta(\boldsymbol{r}_1, \boldsymbol{r}_1\, 0)]^{1/2} [\Delta(\boldsymbol{r}_2, \boldsymbol{r}_2, 0]^{1/2}} \tag{6}$$

where the expression

$$\Delta(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \langle \boldsymbol{E}(\boldsymbol{r}_1, t+\tau) \boldsymbol{E}^*(\boldsymbol{r}_2, t)\rangle, \tag{7}$$

by analogy to scalar theory, will be termed the mutual coherence function[4].

From (4.a), (4.b) and (7) we get

$$\Delta_l^{(E(}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{1}{k_0^2 r_1 r_2} \sum_{m=-l} \sum_{m'=-l}^{l} \left\langle a_{lm}^{(E)}\left(t+\tau - \frac{r_1}{c}\right) a_{lm'}^{(E)*}\left(t - \frac{r_2}{c}\right)\right\rangle \times$$
$$\times (\boldsymbol{X}_{lm}(\theta_1, \varphi_1) \times \boldsymbol{n}_1) \cdot (\boldsymbol{X}_{lm'}^*(\theta_2, \varphi_2) \times \boldsymbol{n}_2) \tag{8.a}$$

for the electric 2^l pole, and

$$\Delta_l^{(M)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{1}{k_0^2 r_1 r_2} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \left\langle a_{lm}^{(M)}\left(t+\tau - \frac{r_1}{c}\right) a_{lm'}^{(M)*}\left(t - \frac{r_2}{c}\right)\right\rangle \times$$
$$\times \boldsymbol{X}_{lm}(\theta_1, \varphi_1) \cdot \boldsymbol{X}_{lm'}(\theta_2, \varphi_2) \tag{8.b}$$

for the magnetic 2^l pole.

[4] It is assumed, unless otherwise stated, that the fields are stationary and ergodic.

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The mean values of products of the amplitudes $a^{(E)}$ and $a^{(M)}$, occuring in (8.a) and (8.b), are also analytic sygnals. With the aid of the Wiener-Khintchine theorem [19] they can be expressed as Fourier transforms of the corresponding spectral densities:

$$\left\langle a_{lm}^{(E}\left(t+\tau-\frac{r_1}{c}\right) a_{lm'}^{(E)*}\left(t-\frac{r_2}{c}\right)\right\rangle = \int_0^\infty d\nu \lim_{T\to\infty}\left\{\frac{a_{lm}^{(E)}(\nu;\, T)a_{lm'}^{(E)*}(\nu;\, T)}{2T}\right\}_{\mathrm{ens}} e^{-2\pi i\nu\left(\tau-\frac{r_1-r_2}{c}\right)} \tag{9.a}$$

$$\left\langle a_{lm}^{(M)}\left(t+\tau-\frac{r_1}{c}\right) a_{lm}^{(M)*}\left(t-\frac{r_2}{c}\right)\right\rangle = \int_0^\infty d\nu \lim_{T\to\infty}\left\{\frac{a_{lm}^{(M)}(\nu;\, T)a_{lm'}^{(M)*}(\nu;\, T)}{2T}\right\}_{\mathrm{ens}} e^{-2\pi i\nu\left(\tau-\frac{r_1-r_2}{c}\right)} \tag{9.b}$$

where $a_{lm}^{(E)}(\nu;\, T)$ and $a_{lm}^{(M)}(\nu;\, T)$ are the Fourier transforms of the truncated functions $a_{lm}^{(E)}(t;T)$ and $a_{lm}^{(M)}(t;\, T)$, see [2]. The symbols $\{\ \}_{\mathrm{ens}}$ indicate an ensemble average; for stationary processes this assures the existence of the limit, as $T\to\infty$. The condition that the radiation is unpolarized and has an isotropic intensity distribution can be fulfilled if we impose the following requirements on the spectral densities of the field (this amounts to taking an average over all possible orientations of a given multipole)

$$\lim_{T\to\infty}\left\{\frac{a_{lm}^{(E)}(\nu;\, T)a_{lm'}^{(E)*}(\nu;\, T)}{2T}\right\}_{\mathrm{ens}} = \delta_{mm'} f_l^{(E)}(\nu), \tag{10.a}$$

$$\lim_{T\to\infty}\left\{\frac{a_{lm}^{(M)}(\nu;\, T)a_{lm'}^{(M)}(\nu;\, T)}{2T}\right\}_{\mathrm{ens}} = \delta_{mm'} f_l^{(M)}(\nu) \tag{10.b}$$

where $f_l^{(E)}(\nu)$ and $f_l^{(M)}(\nu)$ are functions which do not depend on the index m. As can be seen from Eqs (3. a) and (3. b) the explicit form of these functions is affected by the statistics of the current and magnetization distribution. On substituting (10.a) and (10.b) into (8.a) and (8.b), respectively, and making use of the properties of spherical vector harmonics we obtain the following expressions for the mutual coherence functions

$$\Delta_l^{(E)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{1}{k_c^2 r_1 r_2} F_l^{(E)}\left(\tau-\frac{r_1-r_2}{c}\right)\frac{2l+1}{4\pi}\times$$

$$\times\left[\cos\theta\, P_l(\cos\theta) - \sqrt{\frac{(l-1)!}{(l+1)!\, l(l+1)}}\, P_l^1(\cos\theta)\right], \tag{11.a}$$

$$\Delta_l^{(M)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{1}{k_0^2 r_1 r_2} F_l^{(M)}\left(\tau-\frac{r_1-r_2}{c}\right)\frac{2l+1}{4\pi} P_l(\cos\theta) \tag{11.b}$$

where

$$F_l^{(E)}(\tau) = \int_0^\infty f_l^{(E)}(\nu) e^{-2\pi i\nu\tau}\, d\nu, \tag{12.a}$$

$$F_l^{(M)}(\tau) = \int_0^\infty f_l^{(M)}(\nu) e^{-2\pi i\nu\tau} d\nu, \tag{12.b}$$

and θ is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (Fig. 1). $P_l(\cos\theta)$ and $P_l^m(\cos\theta)$ denote as usual the Legendre polynamials and associated Legendre functions (in our case $m = 1$).

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Setting $\mathbf{r}_1$ equal to $\mathbf{r}_2$ and $\tau = 0$ in Eqs (11.a) and (11.b) we obtain expressions proportional to the averaged intensity of multipole radiation. We have

$$\Delta_l^{(E)}(\mathbf{r}_1, \mathbf{r}_1, 0) = \frac{1}{k_0^2 r_1^2} F_l^{(E)}(0) \frac{2l+1}{4\pi}, \tag{13.a}$$

$$\Delta_l^{(M)}(\mathbf{r}_1, \mathbf{r}_1, 0) = \frac{1}{k_0^2 r_1^2} F_l^{(M)}(0) \frac{2l+1}{4\pi} \tag{13.b}$$

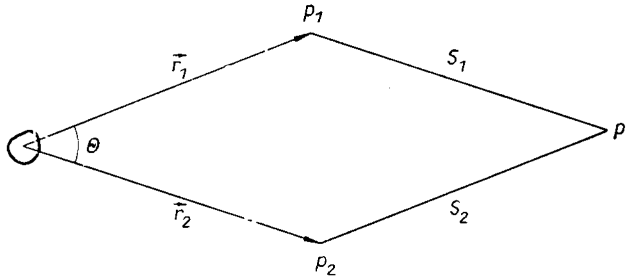

Fig. 1. Illustration of notation

for multipole radiation of electric and magnetic types, respectively. Since there is no dependence on angles, we see that the conditions (10.a) and (10.b) lead evidently to an isotropic intensity distribution. It can also be checked, by the coherence matrix technique [16], that the radiation is completely unpolarized.

From (6) and (11) the electromagnetic degree of coherence is

$$\delta_l^{(E)}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{F_l^{(E)}\left(\tau - \frac{r_1 - r_2}{c}\right)}{F_l^{(E)}(0)} \left[\cos\theta\, P_l(\cos\theta) - \sqrt{\frac{(l-1)!}{(l+1)!\, l(l+1)}} \sin\theta\, P_l^1(\cos\theta)\right], \tag{14.a}$$

$$\delta_l^{(M)}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{F_l^{(M)}\left(\tau - \frac{r_1 - r_2}{c}\right)}{F_l^{(M)}(0)} P_l(\cos\theta) \tag{14.b}$$

for radiation of electric and magnetic types, respectively.

In the above formulas the first term depending on the parameter $\Delta s = \tau - (r_1 - r_2)/c$ describes temporal coherence effects. The second term, which depends on θ and l, describes the multipole structure of the radiation source, and may be said to represent multipole coherence effects. If the field is monochromatic, then regarding the amplitudes $a_{lm}^{(E)}(\nu)$ and $a_{lm}^{(M)}(\nu)$ as random variables and averaging over all possible orientations of a given multipole[5], we find

$$F_l^{(E)}\left(\tau - \frac{r_1 - r_2}{c}\right) = f_l^{(E)}(\nu) e^{-2\pi i\nu\left(\tau - \frac{r_1 - r_2}{c}\right)}, \tag{15.a}$$

$$F_l^{(M)}\left(\tau - \frac{r_1 - r_2}{c}\right) = f_l^{(M)}(\nu) e^{-2\pi i\nu\left(\tau - \frac{r_1 - r_2}{c}\right)}. \tag{15.b}$$

[5] In this case we have example of a field which is not ergodic.

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Hence,

$$\left| F_l^{(E)}\left(\tau - \frac{r_1-r_2}{c}\right) \Big/ F_l^{(E)}(0) \right| = 1, \tag{16.a}$$

$$\left| F_l^{(M)}\left(\tau - \frac{r_1-r_2}{c}\right) \Big/ F_l^{(M)}(0) \right| = 1 \tag{16.b}$$

and the only effect is that of multipole coherence. Similarly, if the optical path difference $\Delta s = 0$, only multipole coherence plays a role. The multipole coherence dependence on the order and type of radiation is presented in the table below

Order \ Type	Electric $\cos\theta P_l(\cos\theta) - \sqrt{\frac{(l-1)!}{(l+1)!\,l(l+1)}}\sin\theta\, P_l^1(\cos\theta)$	Magnetic $P_l(\cos\theta)$
$l = 1$ (dipole)	$\frac{1}{2}(\cos^2\theta+1)$	$\cos\theta$
$l = 2$ (quadrupole)	$\cos^3\theta$	$\frac{1}{2}(3\cos^2\theta-1)$
$l = 3$ (octupole)	$\frac{1}{8}(15\cos^4\theta-6\cos^2\theta-1)$	$\frac{1}{2}(5\cos^3\theta-3\cos\theta)$

3. *Temporal and multipole coherence in the scalar theory*

It is clear that multipole coherence will always be significant in the case of electromagnetic radiation. This results from the fact that spherically-symmetrical solutions of Maxwell's equations do not exist. On the other hand, they do exist when we deal with scalar waves, namely

$$U_0(\boldsymbol{r}, \nu) = a_0(\nu)\frac{e^{ikr}}{r}, \tag{17}$$

which represents the radiation from an isotropic point source. The spectral density corresponding to a quasi-monochromatic spherical wave is

$$g_0(\nu) = \lim_{T\to\infty}\left\{\frac{|a_0(\nu)|^2}{2T}\right\}_{\text{ens}}. \tag{18}$$

This implies that the mutual coherence function has the form

$$\Gamma_0(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{1}{r_1 r_2}\int_0^\infty g_0(\nu)\, e^{-2\pi i\nu\left(\tau - \frac{r_1-r_2}{c}\right)}\, d\nu = \frac{1}{r_1 r_2} G_0\left(\tau - \frac{r_1-r_2}{c}\right). \tag{19}$$

More general solutions of the wave equation can be obtained by differentiation [20]. Let $L_k = \boldsymbol{\alpha}_k \bullet$ grad, then

$$U_N(\boldsymbol{r}, \nu) = L_1 L_2 \dots L_N \left[a_0(\nu) \frac{e^{ikr}}{r} \right] \tag{20}$$

represents a wave from a multipole with N axes $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_N$. For example, in the case of radiation from a dipole with an axis $\boldsymbol{\alpha}$ we have in the far zone

$$U_1(\boldsymbol{r}, \nu) = ik(\boldsymbol{n} \cdot \boldsymbol{\alpha}(\nu)) a_0(\nu) \frac{e^{ikr}}{r}. \tag{21}$$

If we assume that the intensity distribution is isotropic we have to demand

$$\lim_{T \to \infty} \left\{ \frac{\alpha_i(\nu; T)\, \alpha_j^*(\nu; T) |a_0(\nu; T|^2}{2T} \right\}_{\text{ens}} = \frac{1}{3} g_1(\nu) \delta_{ij} \tag{22}$$

where $i, j = 1, 2, 3$ label the Cartesian components of the vector $\boldsymbol{\alpha}$. This yields the mutual coherence function of dipole radiation

$$\Gamma|(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{k_0^2}{r_1 r_2} \frac{\cos\theta}{3} \int_0^\infty g_1(\nu) e^{-2\pi i \nu \left(\tau - \frac{r_1 - r_2}{c}\right)} = \frac{k_0^2 \cos\theta}{3 r_1 r_2} G_1 \left(\tau - \frac{r_1 - r_2}{c}\right) \tag{23}$$

where, as before, k_0 is the mean value of k and θ is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The scalar degree of coherence, defined by

$$\gamma(\boldsymbol{r}_1 \boldsymbol{r}_2, \tau) = \frac{\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)}{[\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_1, 0)]^{1/2} [\Gamma(\boldsymbol{r}_2, \boldsymbol{r}_2, 0)]^{1/2}}. \tag{24}$$

is

$$\gamma_0(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{G_0 \left(\tau - \frac{r_1 - r_2}{c}\right)}{G_0(0)} \tag{25}$$

if the radiation is due to an isotropic point source, and

$$\gamma_1(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \cos\theta \frac{G_1 \left(\tau - \frac{r_1 - r_2}{c}\right)}{G_1(0)} \tag{26}$$

if the radiation is due to a dipole point source. It is seen that only in the first case the multipole coherence effect does not appear.

4. Degree of coherence of the oscillator radiation field

To illustrate the former results we consider the radiation field produced by a damped oscillator. In the dipole approximation the equation of motion of a charged particle of mass m is

$$\ddot{\boldsymbol{x}} + \omega_0^2 \boldsymbol{x} = \frac{2e^2}{3mc^3} \dddot{\boldsymbol{x}} \tag{27}$$

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where the right-hand term represents the radiation damping force. Under the assumption that $\gamma = \frac{2e^2\omega_0^2}{3mc^3} \ll \omega_0$, the equation of motion of the dipole moment $\boldsymbol{p} = e\boldsymbol{x}$ is

$$\ddot{\boldsymbol{p}} + \gamma\dot{\boldsymbol{p}} + \omega_0^2\boldsymbol{p} = 0, \tag{28}$$

whose approximate solution is given by

$$\boldsymbol{p} = \boldsymbol{p}_0 e^{-\frac{\gamma t}{2} - 2\pi i \nu_0 t} \qquad (\omega_0 = 2\pi\nu_0). \tag{29}$$

We assume that excitation starts with a randomly oriented amplitude $\boldsymbol{p}_0$ at $t = 0$. If all orientations are equally probable we have

$$\{\boldsymbol{p}_{0k}\boldsymbol{p}_{0l}^*\}_{\text{ens}} = \frac{1}{3}\,\delta_{kl}|p_0|^2 \tag{30}$$

where p_{0k} denote the components of $\boldsymbol{p}_0$. This assumption assures the isotropy of the intensity distribution.

In complete analogy to former results the electric field of a dipole producing quasi-monochromatic radiation of mean frequency ν_0 is

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{4\pi\nu_0^2}{c^2}\left[\boldsymbol{n}\times\boldsymbol{p}\left(t - \frac{r}{c}\right)\right]\times\boldsymbol{n}. \tag{31}$$

The radiation field of the dipole whose oscillations, according to (29) decay as exp $[-\gamma t/2]$ is not stationary. We should bear in mind, however, that (29) represents the state of a singly excited dipole, whereas the radiation of appreciable intensity arises by successive acts of excitation and emission. If the probability of excitation per unit time is constant we may regard the process as stationary. If, moreover, the period between two successive excitations is much longer than the oscillator lifetime $1/\gamma$ then the time averaged square of the dipole moment, whose excitations occur at $t_1, t_2, \ldots, t_{N_T}$, can be evaluated according to the following scheme

$$\langle|\boldsymbol{p}|^2\rangle = \frac{1}{T}\int_0^T |p(t)|^2\,dt = \frac{n}{N_T}\sum_{k=1}^{N_T}\int_{t_k}^{t_{k+1}} |p(t)|^2 dt \tag{32}$$

where n is the number of excitations per unit time, and N_T is the number of ecitations during a time $T(nT = N_T)$. Owing to the assumption concerning the period between two successive excitations all integrals in (32) may be written as

$$\int_0^\infty |p(t)|^2\,dt = \int_0^\infty |p(\nu)|^2\,d\nu \tag{33}$$

where, by (29),

$$\boldsymbol{p}(\nu) = \int_0^\infty \boldsymbol{p}(t)e^{2\pi i\nu t}\,dt = \boldsymbol{p}_0\frac{1}{2\pi i(\nu_0 - \nu) + \frac{\gamma}{2}}. \tag{34}$$

From the last three formulas we get

$$\langle|\boldsymbol{p}(t)|^2\rangle = n\int\limits_0^\infty |\boldsymbol{p}(t)|^2\,dt = n|\boldsymbol{p}_0|^2\int\limits_0^\infty \frac{d\nu}{4\pi^2(\nu_0-\nu)^2+\frac{\gamma^2}{4}}. \tag{35}$$

This yields the spectral density $P(\nu)$ of the oscillator

$$P(\nu) = \frac{n|p_0|^2}{4\pi^2(\nu_0-\nu)^2+\frac{\gamma^2}{4}} \tag{36}$$

Taking (31) into account we note that in the expression (7) for the mutual coherence function only products of the form $p_{0k}p_{0l}$ appear. These should be averaged according to (30). The necessity of applying both time and ensemble averages results from the non-ergodicity of the process described by (29) whose amplitude is a random variable (*cf.* [19]). Thus the averaged mutual coherence function is

$$\{\Delta(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau)\}_{\text{ens}} = \frac{1}{3}(\cos^2\theta+1)\frac{16\pi^4\nu_0^4 n}{c^4}\int\limits_0^\infty \frac{e^{-2\pi i\nu\left(\tau-\frac{r_1-r_2}{c}\right)}d\nu}{4\pi^2(\nu_0-\nu)^2+\frac{\gamma^2}{4}} \tag{37}$$

where the limits can be extended to $\pm\infty$, since most of the contribution to the integral n (37) comes from the region $\nu\approx\nu_0$. Finally, we obtain

$$\{\Delta(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau)\}_{\text{ens}} = \frac{1}{3}(\cos^2\theta+1)\frac{16\pi^4\nu_0^4}{c^4}\,n e^{-\frac{\gamma}{2}\left|\tau-\frac{r_1-r_2}{c}\right|-2\pi i\nu_0\left(\tau-\frac{r_1-r_2}{c}\right)}, \tag{38}$$

which after normalization yields

$$\delta(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau) = \frac{1}{2}(\cos^2\theta+1)\,e^{-\frac{\gamma}{2}\left|\tau-\frac{r_1-r_2}{c}\right|-2\pi i\nu_0\left(\tau-\frac{r_1-r_2}{c}\right)}. \tag{39}$$

In agreement with the considerations of section 2, the degree of coherence is a product of two factors. The factor $\frac{1}{2}(\cos^2\theta+1)$ characterizes the effect of multipole coherence of electric dipole radiation. The other one is related to the spectral width of radiation and describes the effect of temporal coherence. By considering a definite model of the source we were able to derive the explicit expression for $F_l^{(E)}(\tau)$, $l=1$. In our case

$$F_1^{(E)}(\tau) = \frac{16\pi^4\nu_0^4 n}{c^4}\,e^{-\frac{\gamma}{2}\tau-2\pi i\nu_0\tau}, \tag{40}$$

which agrees with the result obtained by Glauber [21]. The time delay $\Delta t=\frac{2}{\gamma}$ may be called the coherence time.

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5. *Generalization of the van Cittert-Zernike theorem*

Thus far the spatial coherence effects have not been considered. This was possible since, as stressed in section 2, we were concerned with radiation in the far zone, in the case when the angular dimensions of the source could be neglected. In a similar manner, in the scalar formulation of the theory the area of coherence appears to be large if the source is sufficiently far away from the region of observation. We shall now determine the degree of coherence of multipole radiation from an extended incoherent quasi-monochromatic source, taking finite angular dimensions of the source into account.

Imagine the source to be divided into elements dV_1, dV_2, ... centred about points Q_1, Q_2, ... If $\boldsymbol{E}_\alpha(\boldsymbol{r}_1, t)$ and $\boldsymbol{E}_\alpha(\boldsymbol{r}_2, t)$ are the contributions to $\boldsymbol{E}(\boldsymbol{r}_1, t)$ and $\boldsymbol{E}(\boldsymbol{r}_2, t)$ from the element dV_α, then

$$\boldsymbol{E}(\boldsymbol{r}_1, t) = \sum_\alpha \boldsymbol{E}_\alpha(\boldsymbol{r}_1, t), \qquad \boldsymbol{E}(\boldsymbol{r}_2, t) = \sum_\alpha \boldsymbol{E}_\alpha(\boldsymbol{r}_2, t) \tag{41}$$

and the mutual coherence function is given by

$$\begin{aligned} \varDelta(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) &= \langle \boldsymbol{E}(\boldsymbol{r}_1, t+\tau) \cdot \boldsymbol{E}^*(\boldsymbol{r}_2, t)\rangle \\ &= \sum_\alpha \langle \boldsymbol{E}_\alpha(\boldsymbol{r}_1, t+\tau) \cdot \boldsymbol{E}^*_\alpha(\boldsymbol{r}_2, t)\rangle = \sum_\alpha \varDelta_\alpha(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau). \end{aligned} \tag{42}$$

Terms of the type $\langle \boldsymbol{E}_\alpha(\boldsymbol{r}_1, t+\tau)\, \boldsymbol{E}^*_\beta(\boldsymbol{r}_2, t)\rangle$ $(\alpha \neq \beta)$ have been omitted due to the assumed incoherence of the source. If the linear dimensions of the elements dV_α are sufficiently small, and if to each element the conditions (10) apply, we get

$$\begin{aligned} \varDelta^{(E)}_{l\alpha}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) &= \frac{2l+1}{4\pi k_0^2} \frac{1}{r_{1\alpha} r_{2\alpha}} F^{(E)}_{l\alpha}\left(\tau - \frac{r_{1\alpha} - r_{2\alpha}}{c}\right) \times \\ &\times \left[\cos\theta_\alpha P_l(\cos\theta_\alpha) - \sqrt{\frac{(l-1)!}{(l+1)!\,l(l+1)}} \sin\theta_\alpha P^1_l(\cos\theta_\alpha)\right] \end{aligned} \tag{43.a}$$

if the radiation is due to an electric 2^l pole, and

$$\varDelta^{(M)}_{l\alpha}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{2l+1}{4\pi k_0^2} \frac{1}{r_{1\alpha} r_{2\alpha}} F^{(M)}_{l\alpha}\left(\tau - \frac{r_{1\alpha} - r_{2\alpha}}{c}\right) P_l(\cos\theta_\alpha) \tag{43.b}$$

if the radiation is due to a magnetic 2^l pole.

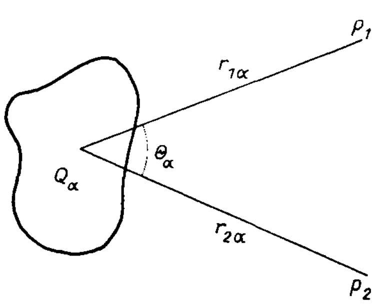

Fig. 2. Illustration of notation

Now $r_{1\alpha}$ and $r_{2\alpha}$ represent the distances of P_1 and P_2 from the source point Q_α (Fig. 2), and θ_α is the angle between $\boldsymbol{r}_{1\alpha}$ and $\boldsymbol{r}_{2\alpha}$. From the formulas (12) it follows that

$$F_{l\alpha}^{(E)}\left(\tau-\frac{r_{1\alpha}-r_{2\alpha}}{c}\right)=\int\limits_0^\infty f_{l\alpha}^{(E)}(\nu)e^{ik(r_{1\alpha}-r_{2\alpha})}e^{-2\pi i\nu\tau}d\nu, \tag{44.a}$$

$$F_{l\alpha}^{(M)}\left(\tau-\frac{r_{1\alpha}-r_{2\alpha}}{c}\right)=\int\limits_0^\infty f_{l\alpha}^{(M)}(\nu)e^{2k(r_{1\alpha}-r_{2\alpha})}e^{-2\pi i\nu\tau}d\nu \tag{44.b}$$

where $f_{1\alpha}^{(E)}$ and $f_{1\alpha}^{(M)}$ are the spectral densities of radiation from the source element dV_α. Suppose that the number of the source elements is so large that we may treat the source as being effectively continuous. If $I(Q,\nu)$ denotes the intensity per unit volume of the source per unit frequency range, then we may replace $f_{1\alpha}^{(E)}$ and $f_{1\alpha}^{(M)}$ by $I_1^{(E)}(Q_\alpha,\nu)dV_\alpha$ and $I_1^{(M)}(Q_\alpha,\nu)dV_\alpha$, respectively. Hence, making use of (42), (43) and (44), we get

$$\Delta_l^{(E)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau)=\frac{2l+1}{4\pi k_0^2}\int\limits_0^\infty d\nu e^{-2\pi i\nu\tau}\int\limits_\nu I_l^{(E)}(Q,\nu)\frac{e^{ik(r_1-r_2)}}{r_1r_2}\times$$
$$\times\left[\cos\theta\, P_l(\cos\theta)-\sqrt{\frac{(l-1)!}{(l+1)!\,l(l+1)}}\sin\theta\, P_1^1(\cos\theta)\right]dV, \tag{45.a}$$

$$\Delta_l^{(M)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau)=\frac{2l+1}{4\pi k_0^2}\int\limits_0^\infty d\nu e^{-2\pi i\nu\tau}\int\limits_\nu I_l^{(M)}(Q,\nu)\frac{e^{ik(r_1-r_2)}}{r_1r_2}P_l(\cos\theta)dV. \tag{45.b}$$

These equations are generalization of the formula for the mutual coherence function of radiation from an extended, incoherent source (*cf.* [16], p. 533, Eq (7)). The generalization concerns both the nature of radiation — the vector formulation is used, and the source structure.

In derivation of (45) it has been assumed that the multipoles are distributed in space. Quite analogous equations can be obtained if the multipoles are distributed across a plane. We only have to replace the volume integrals in (45) by surface integrals over the plane of the source.

The temporal coherence effects can be neglected, provided that the time delay τ is much smaller than the coherence time, which is of the order of the reciprocal of the effective spectral width $\Delta\nu$. In this case the correlations of the field are characterized by the function $\Delta(\boldsymbol{r}_1,\boldsymbol{r}_2,0)$, called the mutual intensity. If we consider the plane source, for example, by virtue of (45) we have

$$\Delta_l^{(E)}(\boldsymbol{r}_1,\boldsymbol{r}_2,0)=\frac{2l+1}{4\pi k_0^2}\int\limits_S I_l^{(E)}(S)\frac{e^{ik_0(r_1-r_2)}}{r_1r_2}\times$$
$$\times\left[\cos\theta P_l(\cos\theta)-\sqrt{\frac{(l-1)!}{(l+1)!\,l(l+1)}}\sin\theta\, P_l^1(\cos\theta)\right]dS, \tag{46.a}$$

$$\Delta_l^{(M)}(\boldsymbol{r}_1,\boldsymbol{r}_2,0)=\frac{2l+1}{4\pi k_0^2}\int\limits_S I_l^{(M)}(S)\frac{e^{ik_0(r_1-r_2)}}{r_1r_2}P_l(\cos\theta)\,dS \tag{46.b}$$

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where

$$I_l^{(E)}(S) = \int_0^\infty I_l^{(E)}(S, \nu) d\nu, \quad I_l^{(M)}(S) = \int_0^\infty I_l^{(M)}(S, \nu) d\nu. \tag{47}$$

The complex degree of coherence $\delta(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)$ is, according to (46) and (6), given by

$$\delta_l^{(E)}(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)$$

$$= \frac{\int_S I_l^{(E)}(S) \frac{e^{ik_0(r_1-r_2)}}{r_1 r_2} \left[\cos\theta\, P_l(\cos\theta) - \sqrt{\frac{(l-1)!}{(l+1)!\, l(l+1)}} \sin\theta\, P_l^1(\cos\theta)\right] dS}{\left[\int \frac{I_l^{(E)}(S)}{r_1^2} dS\right]^{1/2} \left[\int \frac{I_l^{(E)}(S)}{r_2^2} dS\right]^{1/2}}, \tag{48.a}$$

$$\delta_l^{(M)}(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) = \frac{\int_S I_l^{(M)}(S) \frac{e^{ik_0(r_1-r_2)}}{r_1 r_2} P_l(\cos\theta)\, dS}{\left[\int \frac{I_l^{(M)}(S)}{r_1^2} dS\right]^{1/2} \left[\int \frac{I_l^{(M)}(S)}{r_2^2} dS\right]^{1/2}}. \tag{48.b}$$

These equations are generalizations of the van Cittert-Zernike theorem for multipole radiation. If the field is of narrow angular spread, *i. e.* $\theta \to 0$, then for any order of electric or magnetic poles they become

$$\delta(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) = \frac{\int_S I(S) \frac{e^{ik_0(r_1-r_2)}}{r_1 r_2} dS}{\left[\int \frac{I(S)}{r_1^2} dS\right]^{1/2} \left[\int \frac{I(S)}{r_2^2} dS\right]^{1/2}}, \tag{49}$$

which is a mathematical expression of the van Cittert-Zernike theorem in the scalar theory of light.

6. *Concluding remarks*

The investigation presented here is based on some simple assumptions concerning the nature of the radiation source. In particular, it has been assumed that the multipole fields of electric and magnetic type do not appear together, and that the intensity distribution is isotropic. This imposes restrictions on the applicability of our results to a certain class of physical situations. For instance, the common case, when magnetic dipole and electric quadrupole radiation occur jointly, is not included, for then we have to know

$$\lim_{T\to\infty} \left\{ \frac{a_{2m}^{(E)}(\nu; T)\, a_{1m}^{(M)}(\nu; T)}{2T} \right\}_{\text{ens}}.$$

We are able, however, to draw the following conclusions, independent of any specific source characteristic:

1. Partial coherence effects arise, apart from the finiteness of the spectral width and the source size, also from the multipolarity of the source. The effect of multipole coherence indicates that in certain cases we have to admit fields that are not ergodic.

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2. The spatial coherence effects are affected only by the angular size of the source. The temporal coherence effects are described by a function $F\left(\tau - \frac{r_1 - r_2}{c}\right)$, which is determined by the statistics of the current distribution.

3. The van Cittert-Zernike theorem can be generalized for multipole radiation. For fields of narrow angular spread the scalar theory may be employed.

4. The measurements of the degree of coherence yield information about the multipolarity of the source. This confirms the well-known results from wide-angle interference investigations.

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