TRANSFORMATION OF HEAT IN RELATIVISTIC THERMODYNAMICS

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A system enclosed in impenetrable diathermic walls which perform work on and transfer heat to the system in a reversible or irreversible process is studied. Direct calculation of forces from the walls on the system, of work, and of energy leads to the conclusion that heat transferred to the system in all reversible and a number of irreversible processes transforms according to Planck's formula. We ought to redefine the concept of force from the wall in order to get Ott's transformation formula. However, the redefined force is no longer a time average of actual forces from the particles of the wall on the particles of the system and therefore does not have a good physical meaning.

1. *Introduction*

Relativistic thermodynamics deals with the thermodynamic processes in rapidly moving bodies. The body or its parts may be accelerated but, in the special theory of relativity, we must look at it from an inertial system of reference. The principles of relativistic thermodynamics were formulated early in the history of the special theory of relativity by Planck and other autors [1]–[3]. Planck's investigation rests on a supposition that the first and the second law of thermodynamics are valid in every inertial system in the form we are accustomed to from classical thermodynamics. It is well known that the demand of form invariance of physical laws with respect to the transformations from one inertial system to another enforces the transformation properties of the quantities which stand in the laws in question. In this manner, we get *e.g.* the transformation properties of electric and magnetic fields from the demand of form invariance of Maxwell equations. In a similar manner, Planck got the transformation property of heat transferred to the body from the demand of the invariance of the first law of thermodynamics, and the transformation property of absolute temperature from the demand of the invariance of the second law of thermodynamics.

The simplest type of thermodynamic process we study in the theory of relativity has the following characteristic development. At the beginning of the process, a thermodynamic

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system is in a state of thermodynamic equilibrium at rest in an inertial system I^0. To be more specific, the system I^0 is the rest system of mass of each element of the thermodynamic system. In the process, which may be reversible or irreversible, the thermodynamic system moves in general in I^0. However, the process ends in such a state that the thermodynamic system is again at rest in the initial system I^0 and settled in thermodynamic equilibrium.

According to the first law of thermodynamics, the total energy H^0 of the thermodynamic system in I^0 is a function of its equilibrium state. If, in a process, the surroundings perform the work ΔA^0 on the system and transfer the heat ΔQ^0 to the system, the total energy H^0 of the system increases by

$$\Delta H^0 = \Delta Q^0 + \Delta A^0. \tag{1}$$

We suppose now, in accordance with the principle of relativity, that the law of the same form holds also in an inertial system I, in which the thermodynamic system before and after the process moves with the velocity w,

$$\Delta H = \Delta Q + \Delta A. \tag{2}$$

If we know how the total energy of the system changed and what mechanical work was performed by the external forces in the process, we can use the first law of thermodynamics as a definition of transferred heat. It is sufficient to say that heat transferred in a thermodynamic process to the system is defined as that increase of the total energy of the system which is not due to the mechanical work performed by external forces on the system. This definition may be used in the new inertial system I as well as in the old system I^0. If we know independently how the total energy and the work of external forces transform when passing from I^0 to I, we find therefrom the transformation law of heat transferred to the system in the process.

The energy of the system and the forces acting on the system are given in the theory of relativity by the energy-momentum tensor of the system. When we know this tensor throughout the process, we have no principal difficulties in calculating the numerical value of the energy of the system in I at the moment t, though it may be difficult to express the energy as a function of macroscopic state. From the energy momentum tensor, we can also determine the work dA performed on the system by external forces in an infinitesimal interval of time between the moments t and $t+dt$. It is therefore possible to define not only the heat transferred to the system in a transition from one equilibrium state to another, but even the heat transferred to the system in a small time interval dt,

$$dQ = dH - dA. \tag{3}$$

Moreover; if the system is enclosed by diathermic walls and the only external forces acting on the system are just the forces from the walls, it is possible to define that heat δQ, which is transferred to the system in an interval of time dt through an element $d\Sigma$ of the wall. Indeed, the energy momentum tensor determines not only the energy, but also the energy current, *i.e.* what part of energy δH is transferred to the system in an interval of time dt through the surface element $d\Sigma$. It determines also the force $\delta \boldsymbol{F}$, which the surface element exerts on the system and the work δA of this force in the interval of time dt. The heat δQ may be therefore defined by the formula

$$\delta Q = \delta H - \delta A. \tag{4}$$

We shall prove in Section 5 that the transformation properties of the local quantity δQ are simpler than the transformation properties of the global quantities dQ and ΔQ. It is important, however, that for the definition of all these quantities it is sufficient to know the energy momentum tensor of the system during the process and nothing more.

The second law of thermodynamics assumes the existence of a function S^0 of the thermodynamic state called entropy. When we change an equilibrium state of the system of absolute temperature T^0 reversibly, transferring to the system an infinitesimal amount of heat dQ^0_{rev}, the entropy of the system increases by

$$dS^0 = \frac{dQ^0_{\text{rev}}}{T^0}. \tag{5}$$

We again suppose that in the system I, with respect to which the thermodynamic system moves with the velocity w, the second law of thermodynamics has the same form as in T^0,

$$dS = \frac{dQ_{\text{rev}}}{T}. \tag{6}$$

By an argument we shall repeat in Section 3, Planck derived the transformation law

$$dQ = dQ^0 \sqrt{1-w^2} \tag{7}$$

for the heat transferred to a system in a short time interval in a reversible process. Because the system, which is initially at rest in I^0 and then adiabatically and gently accelerated till it comes to rest in a new system I, preserves its entropy by (6) and, moreover, is in the same state with respect to I at the end of the process as it was with respect to I^0 at the beginning of the process, Planck concluded that entropy of the system is a relativistic invariant,

$$S = S^0. \tag{8}$$

From (8) and the second law of thermodynamics (3) we get Planck's transformation law of absolute temperature,

$$T = T^0 \sqrt{1-w^2}. \tag{9}$$

In the year 1963, H. Ott [4] doubted the correctness of that part of Planck's argument, which leads to the transformation law of heat (7). The nature of Ott's objections will be explained in Section 3. Instead of (7), Ott proposed the transformation law

$$dQ_O = dQ^0/\sqrt{1-w^2}. \tag{10}$$

From it, using the rest of the Planck's method, the transformation law of absolute temperature follows,

$$T_O = T^0/\sqrt{1-w^2}. \tag{11}$$

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Though five years has passed since the publication of Ott's paper, the situation still does not seem to be completely clarified. A number of mutually contradicting papers appeared, with different proposals for definitions and transformation properties of both thermodynamic and purely mechanical quantities [5]–[20]. If physical law were to be determined by vote, we should come, by comparing the results of the papers [4]–[20], to the ratio 9 : 6 for Ott's proposal. This is a majority which, in a democratic country, suffices to pass the bill, but scarcely to change the constitution. If we take the standpoint that physical laws should be changed after a clear discussion rather than by voting, the situation becomes even more complicated.

From the different methods proposed to solve the problem, the phenomenological approach chosen by Møller in his paper [17] seems to me most convincing. Møller studies in details the plain case of thermodynamic processes in perfect fluid enclosed in a vessel with movable pistons, into which heat flows through the walls. The work of forces which the walls and the pistons exert on the fluid is calculated directly. All assumptions are clearly stated, all concepts are defined, and the calculations proceed carefully step by step. Using this method, Møller comes to the conclusion that Ott's transformation law for temperature is correct and Ott's transformation low for heat is correct at least for all reversible processes, while for irreversible processes it must be replaced by a more general formula.

It is just the fact that all assumptions are clearly formulated in Møller's paper, which enables us to think about their justification. Checking Møller's proof, it seems that only one assumption deserves a closer examination. It is the assumption that we are allowed, when calculating the forces exerted by the walls on the fluid during a reversible process, to replace the actual fluid, which receives the heat from the walls, by an ideal fluid, which is in an equilibrium state at rest with respect to the walls and acts on them only by means of a hydrostatic pressure. Indeed, we shall show that this assumption simplifies the situation to such an extent that it neglects just the effects which may bridge the gap between Planck's and Ott's formulae.

Therefore, we propose in this paper to reexamine once more Moller's procedure. In Section 2, we study the energy momentum tensors, suitable for the description of a fluid with a flow of heat. In Section 3, we compare Planck's and Ott's conceptions in the physical situation studied by Møller, namely, for the fluid enclosed in a container, which reversibly performs work on and transfers heat to the fluid. In Section 4, we follow the Møller's proof in the simple case, that the walls of the container are fixed in I^0 and heat flows reversibly to the fluid. We point out that by replacing Møller's assumption by a more detailed picture of the behaviour of the fluid which conducts heat (the picture we obtained in Section 2), we arrive at old Planck's formula (7). In this way, the physical nature of the effect which underlies Planck's ideas, becomes apparent. In Section 5, we generalize our results to an arbitrary medium enclosed in moving diathermic walls. We show that Planck's formula

$$\delta Q = \delta Q^\circ \sqrt{1-v^2}$$

for the heat δQ transferred through an element of the wall to the medium in a system I, and in the instantaneous rest system I^0 of the wall, respectively, holds for all reversible and irreversible processes; $\boldsymbol{v}$ is the velocity of an element of the wall in I. Further, Planck's transformation formula

$$\Delta Q = \Delta Q^0 \sqrt{1-w^2} \tag{12}$$

for the heat ΔQ transferred to the medium in a process leading from one equilibrium state to another equilibrium state, holds for all reversible or irreversible processes, with the exception of those irreversible processes, in which heat is transferred to the medium in some stage of the process through the wall moving in the system I^0 (the global rest system of the medium before and after the process) rapidly in the direction parallel to the motion of I^0 with respect to I. For the absolute temperature, Planck's formula (9) holds. In Section 6, we investigate whether the whole difference between Planck's and Ott's formulae is not just a matter of definition of force. Indeed, we shall show that the force acting from the walls on the medium may be defined so that the first law of thermodynamics leads to Ott's formula for transferred heat. However, this force has only a formal meaning. The actual force, which is the time average of all microscopic forces acting from the particles of the wall on the particles of the medium by means of elastic collisions, leads unambiguously to Planck's formula.

Greek indices run in this paper through the values 1, 2, 3, Latin indices through the values 1, 2, 3, 4. The velocity of light equals unity by definition, $c = 1$. The local rest system of the mass of the medium is denoted as I^*, the local rest system of an element of the wall (and the rest system of particles) is denoted by I°, and the global rest system of the thermodynamic system is denoted by I^0. For the quantities related to an element of the wall, the symbol δ is used, for quantities related to the wall as a whole, the differential d is used, and for quantities relating to the transition from one equilibrium state to another, the symbol Δ is used.

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2. *Fluids conducting heat*

The mechanical properties of a medium, which at a time t occupies a place x_ι in an inertial system I, are described by the symmetric energy-momentum tensor $T_{ik}(x_e)$. When we write

$$T_{44} = -h,\ T_{4\iota} = iS_\iota,\ T_{\iota 4} = ig_\iota,$$

then h means the energy density, $\boldsymbol{S}$ the energy current density, $\boldsymbol{g}$ the momentum density, and $T_{\iota\varkappa}$ the momentum current density in the medium. We assume that there is always an inertial system I^*, in which the energy current vanishes at the place and time considered,

$$\boldsymbol{S}^* = 0. \tag{13}$$

We call this system the local rest system of mass. The velocity $\boldsymbol{u}$ of the local rest system of mass I^* with respect to an inertial system I may be taken as the macroscopic velocity of matter; the corresponding four-velocity is denoted by U_i. The energy-momentum tensor can be decomposed into two parts,

$$T_{ik} = h^* U_i U_k + S_{ik}, \tag{14}$$

the kinetic tensor, and the stress tensor S_{ik}. There is

$$h^* = T_{ik}U_iU_k,\ S_{ik} = (\delta_{il}+U_iU_l)(\delta_{km}+U_kU_m)T_{lm}$$

$$S_{ik}U_k = 0. \tag{15}$$

In the local rest system of mass, only the 44-component of the kinetic tensor and the space components $S^*_{\iota\varkappa}$ of the stress tensor do not vanish.

The simplest mechanical medium is perfect fluid, characterized by the fact that $S^*_{\iota\varkappa}$ is isotropic in I^*,

$$S^*_{\iota\varkappa} = p^*\delta_{\iota\varkappa}.$$

In any other system we get

$$T_{ik} = (h^*+p)U_iU_k+p\delta_{ik}, \quad p = p^*. \tag{16}$$

Energy density and momentum density in a moving fluid are

$$h = \frac{h^*+p^*u^2}{1-u^2}, \tag{17}$$

$$\boldsymbol{g} = (h+p)\boldsymbol{u} = \frac{h^*+p^*}{1-u^2}\boldsymbol{u}. \tag{18}$$

If the fluid is composed of one kind of particles, which during the motion and collisions preserve their rest masses, we call it simple. The number of particles μ per unit volume and the current density $\boldsymbol{\mu}$ of the number of particles may be gathered into a four-vector

$$\mu_i = (\mu_\iota,\ i\mu), \tag{19}$$

the particle four-current. Similarly as the rest system of mass I^* is introduced by (13), the rest system of particles is introduced by

$$\boldsymbol{\mu}^\circ = 0. \tag{20}$$

The velocity $\boldsymbol{v}$ of I° with respect to I may be taken as the macroscopic velocity of particles; the corresponding four-velocity is denoted by V_i. It follows that $\mu_i = \mu^\circ V_i$. For a heat non-conducting fluid,

$$U_i = V_i,\ \mu_i = \mu^* U_i \tag{21}$$

If we want to describe a fluid which conducts heat, at least one of the assumptions (16), (21) must be abolished. Let us try to preserve (16). To achieve this, we must suppose that the three-dimensional tensor $S^*_{\iota\varkappa}$ is isotropic, even if heat flows through the fluid. The energy-momentum tensor has then still the simple structure of the energy-momentum tensor (16) of the perfect fluid. Of course, the particle current cannot vanish in the rest system of mass I^*, so that

$$\mu^*_i = (u^*_\iota,\ i\mu^*), \quad \mu^*_\iota \neq 0.$$

In a system I,

$$\mu_i = \mu^* U_i+\nu_i, \tag{22}$$

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where

$$\nu^*_i = (\mu^*_\iota, 0),$$

$$\nu_i = (\delta_{ik} + U_i U_k)\mu_k, \ \nu_i U_i = 0. \tag{23}$$

On the other hand, when we pass from the rest system of mass I^* back to the rest system of particles I°, we find that the energy current density $\boldsymbol{S}^\circ$ does not vanish. The vector $\boldsymbol{S}^\circ$ describes in I° just the flow of heat.

The heat-conducting (non-viscous) perfect fluid is characterized in this way *e.g.* by Landau and Lifshitz [21]. There is yet another approach used in the literature, originally proposed by Eckart [22], and elaborated by a number of authors [23]–[26]. The basic assumption of this proposal is that the space components of the energy momentum tensor of heat-conducting fluid is not isotropic in the rest system of mass, but in the rest system of particles,

$$T^\circ_{\iota\varkappa} = p^\circ \delta_{\iota\varkappa}.$$

The mixed components $T^\circ_{4\iota} = i\delta^\circ_\iota$ do not vanish, as they describe the flow of heat. The energy momentum tensor in an arbitrary system I has then the form

$$T_{ik} = (h^\circ + p)V_i V_k + p\delta_{ik} + T_i V_k + T_k V_i, \tag{24}$$

where

$$T^\circ_i = (S^\circ_\iota, 0), p = p^\circ,$$

$$h^\circ = T_{ik} V_i V_k, \quad T_i = (\delta_{ik} + V_i V_k) T_{kl} V_l, \quad T_i V_i = 0.$$

Coming back from I° to I^*, we see that the transformed energy-momentum tensor (24) does not have the simple structure (16). The tensor (16) is therefore not physically equivalent to the tensor (24). In this paper, we are not directly interested in the question which tensor better describes the heat-conducting relativistic fluid. It may be argued that the methods of statistical physics lead naturally to the Landau–Lifshitz tensor [27]. The Landau–Lifshitz tensor is further very well suited to the intuitive arguments, because it has a simple formal structure (16). We will therefore use it for the explanation of the physical origin of the Planck's force in Section 4. This assumption is not vital for the general validity of our results. Indeed, we shall show in Section 5 that it does not matter which particular form of the energy-momentum tensor is used when determining the transformation properties of heat and temperature. The transformation laws are therefore valid irrespective of the fact whether the heat-conducting fluid is governed by Landau–Lifshitz or Eckart tensor.

The heat-conducting fluid need not differ from the adiabatic perfect fluid in the structure of the energy momentum tensor (16), but it must then differ in the structure of particle four-current, as we see when comparing (22) with (21). If the fluid is not simple, an individual particle four-current must be introduced for each kind of particles. To arrive at the concept of a rest system of particles, the total particle current must be defined by summing the currents of particles of different kinds multiplied by suitable weights. The natural weight of a kind of particles is their rest mass [24]. However, for our own purpose it is better to

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avoid the introduction of particle four-current altogether, substituting for it the concept of impenetrable wall.

We may imagine that our system is enclosed in walls, which neither let the particles of the surroundings in, nor the particles of the system out, and which neither absorb the particles of the system nor react with them. A typical interaction of a particle of the system with the wall is an elastic collision, in which the particle preserves its rest mass. A wall having these properties will be called impenetrable. The impenetrable wall may of course mediate the flow of heat from the surroundings to the system by means of elastic collisions, having the properties of diathermic wall. The walls can be also macroscopically movable and therefore perform work on the system.

Because each particle which falls on the wall must be reflected, the normal component of four-current of each kind of particles must vanish at the wall. For this reason, the concept of impenetrable wall replaces the concept of particle four-current and enables us to discern what is the flow of heat and what is not. The concept of impenetrable adiabatic or diatermic wall is very intuitive and is therefore frequently used as a primitive concept in elementary classical thermodynamics. It plays a similar methodological role when defining heat and temperature in relativistic thermodynamics.

3. Planck's and Ott's transformation formulae

Let us take for our thermodynamic system a perfect fluid enclosed in a cylindrical vessel with a movable piston. Before the process, the vessel is thermally and mechanically isolated and the system is in thermodynamic equilibrium at rest with respect to I^0. The axis of the cylinder is chosen for the x^0 axis of our system of coordinates. At the moment t_1^0, we bring the diathermic cylindrical walls of the vessel in contact with a heat reservoir, which has only a slightly higher temperature than the fluid, and slowly push the piston. The heat begins to flow reversibly into the fluid and the piston reversibly performs work. In the classical argument it is supposed that the fluid passes through a succession of equilibrium states. The state at the time t^0 is characterized in I^0 by the fact that it has a certain volume V^0 and throughout this volume a constant value of pressure p^0 and temperature T^0. In an interval dt^0, the volume changes by dV^0, the pressure by dp^0, and the temperature by dT^0. The piston performs thereby the work $dA^0 = -p^0dV^0$ against the pressure p^0, the heat dQ^0 is transferred to the fluid through the cylindrical wall, and the total energy of the fluid increases by dH^0. The supply of heat and the motion of the piston is stopped at the moment t_2^0, and the process ends.

Let us study the process from the point of view of an observer in I, where the vessel moves uniformly with the velocity $\boldsymbol{v} = (v, 0, 0)$. The bottom a of the vessel and the piston b have the same area in I as in I^0, while the length of the cylinder is subject to the Lorentz contraction, so that

$$V = V^0 \sqrt{1-v^2}. \tag{25}$$

When we integrate the energy density (17) and the momentum density (18) over the volume V

of the vessel at a fixed time t in I, we get the transformation formulae for total energy H and total momentum $\boldsymbol{G}$,

$$H = \frac{H^0 + v^2 p^0 V^0}{\sqrt{1-v^2}} \tag{26}$$

$$\boldsymbol{G} = \frac{H^0 + p^0 V^0}{\sqrt{1-v^2}} \boldsymbol{v}. \tag{27}$$

We assumed that the process is reversible so that the values of h^*, p^* at different points in the vessel at times $t^0 = (t+vx)/\sqrt{1-v^2}$ can be replaced by the value of these quantities at the bottom of the vessel. Integration is thereby reduced to the multiplication by the volume (25). Let us note that the total energy and the total momentum of the fluid enclosed in the vessel do not transform like the components of a four-vector.

Planck's argument runs then as follows.

Although the velocity $\boldsymbol{v}$ of the system remains constant from t to $t+dt$, the total momentum ncreases by

$$d\boldsymbol{G} = \frac{d(H^0 + p^0 V^0)}{\sqrt{1-v^2}} \boldsymbol{v}. \tag{28}$$

It means that the walls of the vessel must exert the force $\boldsymbol{F} = d\boldsymbol{G}/dt$ on the fluid. This force is necessary to maintain the constant velocity of the fluid. The walls of the vessel move collectively with the velocity $\boldsymbol{v}$, so that this force performs the work

$$\boldsymbol{v} \cdot d\boldsymbol{G} \tag{29}$$

in the time dt. This work must be added to the work

$$-pdV = -p^0 dV^0 \sqrt{1-v^2} = dA^0 \sqrt{1-v^2} \tag{30}$$

of the piston (moving reversibly with respect to the remaining walls) in order to get the total work of the walls on the fluid,

$$dA = dA^0 \sqrt{1-v^2} + \boldsymbol{v} \cdot d\boldsymbol{G}. \tag{31}$$

Substituting into the last equation the expression (28) for dG, we get

$$dA = dA^0 \sqrt{1-v^2} + \frac{d(p^0 V^0)}{\sqrt{1-v^2}} v^2 + \frac{dH^0}{\sqrt{1-v^2}} v^2. \tag{32}$$

The increase in the total energy is obtained by differentiating (26). Now we are able to calculate the heat dQ transferred to the system during the time dt from the first law of thermodynamics,

$$dQ = dH - dA$$

$$= dH^0 \sqrt{1-v^2} - dA^0 \sqrt{1-v^2} = dQ^0 \sqrt{1-v^2}.$$

This is Planck's transformation formula for heat.

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Ott's objection to Planck's argument is directed against Planck's assertion that the total force $d\boldsymbol{G}/dt$ is necessary for the preservation of uniform velocity of the fluid in I. Indeed, the formula (28) may be written as

$$d\boldsymbol{G} = \frac{dQ^0}{\sqrt{1-v^2}}\boldsymbol{v} + \frac{V^0 dp^0}{\sqrt{1-v^2}}\boldsymbol{v}. \tag{33}$$

Ott imagines that the reversible transfer of heat into the system is equivalent to an accretion of a particle of rest mass dQ^0 to the system. With respect to I^0 this particle moves with an infinitesimal velocity. With respect to I, it moves with the velocity $\boldsymbol{v}$ and has the momentum $dQ^0/\sqrt{1-v^2}\cdot\boldsymbol{v}$. The momentum is carried to the system by the particle and it is therefore not necessary that the walls should act on the system by the force $dQ^0/dt\sqrt{1-v^2}\cdot\boldsymbol{v}$ in order to create the momentum. We will return to this basic assumption in Section 6.

According to Ott's ideas, the walls exert on the fluid only the force

$$\boldsymbol{F}_O = \frac{d\boldsymbol{G}}{dt} - \frac{dQ^0/dt}{\sqrt{1-v^2}}\boldsymbol{v} = \frac{V^0 dp^0/dt}{\sqrt{1-v^2}}\boldsymbol{v} \tag{34}$$

and this force performs the work

$$dA_O = dA - \frac{dQ^0}{\sqrt{1-v^2}}v^2 = \frac{V^0 dp^0}{\sqrt{1-v^2}}v^2 \tag{35}$$

during the time dt.

Calculating the heat dQ_O by means of the first law of thermodynamics, we get

$$dQ_O = dH - dA_O = dH - dA + \frac{dQ^0}{\sqrt{1-v^2}}v^2 = dQ + \frac{dQ^0}{\sqrt{1-v^2}}v^2$$

$$= dQ^0\sqrt{1-v^2} + \frac{dQ^0}{\sqrt{1-v^2}}v^2 = \frac{dQ^0}{\sqrt{1-v^2}}.$$

This is Ott's transformation formula.

The difference between Planck's and Ott's transformation formulae is due to the fact that Planck includes the work of the force

$$\boldsymbol{F}_P = \frac{dQ^0/dt}{\sqrt{1-v^2}}\boldsymbol{v} = \frac{dQ^0}{dt^0}\boldsymbol{v}, \tag{36}$$

the reality of which Ott denies, into the total work of the walls. Ott uses the term "Führungskraft" for this type of force; however, we will call it simply Planck's force.

In order to decide whose ideas about forces are right, it is best to calculate the forces directly. This task was performed by Møller [17]. By a detailed analysis of the described reversible process, Møller concluded that Ott's formulae (34) and (35) for the force and the work are correct. In this way, the problem seemed to be solved in favour of Ott's transformation formula (10) for heat transferred to the system reversibly by conduction.

However, as we mentioned in the *Introduction*, Møller's proof used a simplifying assumption, which from the very beginning smooths out the dynamical effects leading to

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Planck's force. For this reason, we repeat the calculations once more without Møller's assumption. Not to render our work unnecessarily difficult, we shall restrict ourselves at first to the study of the vessel the walls of which are fixed in I^0, the cylindrical walls being diathermic and the bottoms adiabatic. The calculations are then easier to follow and the physical mechanism giving rise to Planck's force becomes apparent. In Section 5, we shall generalise our procedure not only to the case that adiabatic piston reversibly compresses the perfect fluid, but directly to the case of an arbitrary medium characterized by an unspecified energy momentum tensor and enclosed by rapidly deforming diathermic walls through which heat flows in irreversibly.

4. On the dynamical origin of Planck's force

Let us have the vessel fixed in $I^0 \equiv I^\circ$. The heat flows in reversibly and symmetrically through the cylindrical wall. Møller assumes that the fluid in the vessel is in equilibrium at each instant t° in I° and exerts the pressure $p(t^\circ)$ on the walls. The walls are standing in I° and cannot perform the work, so that $dH^\circ = dQ^\circ$. In I, the vessel moves with the velocity $\boldsymbol{v}$ in the direction of its axis. Because the pressure is a relativistic scalar, the moving fluid acts on the moving walls by a force which is again perpendicular to walls in I. The forces from the cylindrical wall mutually cancel by virtue of the symmetry. Moreover, the cylindrical wall cannot perform work in I, because its velocity $\boldsymbol{v}$ is perpendicular to the force. It remains to determine the contribution from the bottoms. At the time t, the fluid exerts the pressure $p(x_{(a)}, t)$ on the bottom a, and because the surface area Σ of the bottom is the same in I as in I°, the bottom acts on the fluid by the force

$$\boldsymbol{F}_{(a)} = p(x_{(a)}, t)\, \Sigma^\circ\, (1, 0, 0).$$

In the same fashion, we get

$$\boldsymbol{F}_{(b)} = p(x_{(b)}, t) \Sigma^\circ\, (-1, 0, 0)$$

for the second bottom b. In I° at t°, the pressure on a has the same value as the pressure on b. In I, however, the pressures $p(x_{(a)}, t)$ and $p(x_{(b)}, t)$ differ by virtue of the relativity of simultaneity. There is

$$p(x_{(b)}, t) - p(x_{(a)}, t) = p^\circ(\mathring{x}_{(b)}, \mathring{t}_{(b)}) - p^\circ(\mathring{x}_{(a)}, \mathring{t}_{(a)}) = \frac{dp^\circ(\mathring{t}_{(a)})}{d\mathring{t}_{(a)}} (\mathring{t}_{(b)} - \mathring{t}_{(a)}) = \frac{d\mathring{p}_{(a)}}{d\mathring{t}_{(a)}}\, l^\circ v,$$

l° being the distance between the bottoms in I°. We neglected the higher terms in the Taylor expansion, because the pressure changes reversibly in I°. For the total force from the walls on the fluid in I, we get

$$\boldsymbol{F}_M = \boldsymbol{F}_{(a)} + \boldsymbol{F}_{(b)} = \frac{d\mathring{p}_{(a)}}{d\mathring{t}_{(a)}} V^\circ \boldsymbol{v} = V^\circ \frac{d\mathring{p}_{(a)}}{dt_{(a)}} \frac{\boldsymbol{v}}{\sqrt{1-v^2}}. \tag{37}$$

This corresponds to Ott's formula (34). There is no place for Planck's force (36) in Møller's picture.

Møller's arguments rests on the assumption that the fluid may be replaced in I° at each instant t° by a fluid in equilibrium, which is at rest with respect to the walls and exerts the pressure p° on them. The pressure p° gradually increases and also the total energy $H^\circ = \int h^\circ dV^\circ$ of the fluid gradually increases by the compressional energy. However, we can easily check that the last assertion, strictly speaking, contradicts the law of conservation of energy. Let us suppose that the heat-conducting fluid is described by the Landau–Lifshitz tensor, which has the same structure as the energy momentum tensor of adiabatic perfect fluid. If the fluid is at rest, then

$$T^\circ_{44} = -h^\circ = -h^*, \qquad S^\circ_\iota = -iT^\circ_{4\iota} = 0,$$

and in accordance with the law of conservation of energy,

$$\frac{\partial h^\circ}{\partial t} = 0, \qquad H^\circ = \text{const}.$$

In order that H° may increase with time, the fluid at the walls must have a non-vanishing velocity $\boldsymbol{u}^\circ$ in I°, however small this velocity is. The velocity $\boldsymbol{u}^\circ$ is of course cylindrically symmetric and, if energy H° increases, directed to the axis. The existence of $\boldsymbol{u}^\circ$ is easy to understand. The system I° is not only the global rest system of the fluid, $\boldsymbol{G}^\circ = 0$, but primarily the rest system of the walls. The cylidrical wall represents the rest system of particles of the fluid. However, because the heat flows in through this wall, there is a flow of energy at the wall, which means that the local rest system of mass moves with respect to the wall (to the local rest system of the particles). The existence of $\boldsymbol{u}^\circ$ thus expresses the plain fact that heat flows from the cylindrical wall to the axis.

However, when we look at the same situation from the point of view of I, where the vessel moves with the velocity $\boldsymbol{v}$ parallel to its axis, the field of velocities u changes. According to the addition law of velocities, $\boldsymbol{u}$ has both a component perpendicular to the wall and a component parallel to $\boldsymbol{v}$,

$$\boldsymbol{u} = u^\circ \sqrt{1-v^2}\,\boldsymbol{n} + \boldsymbol{v}. \tag{38}$$

And it is just this aberation of velocity which leads in a moving vessel to the rise of Planck's force from the cylindrical wall on the fluid. Indeed, the velocity u° is arbitrarily small in comparison with the velocity of light, because heat flows reversibly into the vessel, and we may neglect the terms $(u^\circ)^2$ against unity, *e.g.*

$$1-u^2 = (1-v^2)(1-(u^\circ)^2) \approx 1-v^2. \tag{39}$$

But, on the other hand, we must leave the terms linear in u° in the expressions for force and work, because the field of velocities must exist for a sufficiently long time in order that the reversible supply of heat may lead to an observable increase of the total energy of the fluid at all. In other words, the limit $u^\circ \to 0$, $t_2^\circ - t_1^\circ \to \infty$ may be taken after the calculation of the total work of the walls in the process, not before this calculation, what Møller implicitly does in his assumption.

Let us find at first a general expression for the force, which acts in I from the impenetrable surface $d\Sigma$ of the velocity $\boldsymbol{v}$ on the medium described by the energy momentum tensor T_{ik} on that side of $d\Sigma$, where the unit normal $\boldsymbol{n}$ points to. We understand by force that part of momentum given by the surface to the medium per unit time which is not carried to the medium by penetrating particles. Because our surface is impenetrable for the particles, the force from the surface on the medium is simply equal to the momentum which passes through the surface in the direction of $\boldsymbol{n}$ per unit time.

If the surface rests in I, the momentum

$$T_{\iota\varkappa} n_\varkappa d\Sigma$$

passes through it in the direction $\boldsymbol{n}$ per unit time. If, on the other hand, the surface moves with respect to a medium with vanishing flow of momentum, the momentum

$$g_\iota v_\varkappa n_\varkappa d\Sigma$$

which was located in the volume $\boldsymbol{v}\cdot\boldsymbol{n}d\Sigma$ escapes through the surface per unit time in the direction $-\boldsymbol{n}$. The force from the moving surface on the medium with the flow of momentum is composed from both terms,

$$\delta F_\iota = (T_{\iota\varkappa} - g_\iota v_\varkappa) n_\varkappa d\Sigma. \tag{40}$$

The Landau–Lifshitz tensor gives in I

$$T_{44} = -\frac{h^*+p}{1-u^2} + p, \qquad S_\iota = g_\iota = (h^*+p)\frac{u_\iota}{1-u^2},$$

$$T_{\iota\varkappa} = (h^*+p)\frac{u_\iota u_\varkappa}{1-u^2} + p\delta_{\iota\varkappa}.$$

The corresponding expressions in I° are obtained by putting $v = 0$, $u_\iota = \overset{\circ}{u}_\iota$.

At first we determine the force from the bottoms on the fluid in I. There is

$$\delta F_\iota = p n_\iota d\Sigma,$$

because the additional contributions containing $\boldsymbol{u}$ from $T_{\iota\varkappa}$ and g_ι in (40) just cancel. The further argument is the same as Møller used and we get the result (37).

Much more interesting is the calculation of the force from the cylindrical wall. The second term in (40) vanishes because of $\boldsymbol{n} \perp \boldsymbol{v}$, and we get

$$\delta F_\iota = \left(p + (h^*+p)(u^\circ)^2\frac{1-v^2}{1-u^2}\right) d\Sigma\, n_\iota + \frac{h^*+p}{1-(u^\circ)^2}\, u^\circ \frac{1}{\sqrt{1-v^2}}\, d\Sigma\, v_\iota .$$

The motion of the fluid with respect to the vessel has a consequence that the normal force from the wall is not exactly given by the pressure p, not even in the system I°. However, the normal force does not interest us too much, because the resulting force vanishes due to the symmetry and the work vanishes due to $\boldsymbol{n}\perp\boldsymbol{v}$. What is important is the fact that the new component of force grew in the direction of $\boldsymbol{v}$. This component arises because the velocity $\boldsymbol{u}$ has in I both a component in the direction of $\boldsymbol{v}$ and one in the direction of $\boldsymbol{n}$, so that the projection $\boldsymbol{g}\cdot\boldsymbol{v}/v$ of the momentum flows into the vessel and gives an impulse to the fluid. Because of $d\Sigma = d\Sigma^\circ\sqrt{1-v^2}$, we get for this tangential component of force

$$\delta \boldsymbol{F}^{(v)} = \frac{h^*+p}{1-(u^\circ)^2}\, u^\circ d\Sigma^\circ \boldsymbol{v} = \boldsymbol{S}^\circ \cdot \boldsymbol{n}^\circ d\Sigma^\circ \boldsymbol{v}. \tag{41}$$

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However, the expression $\boldsymbol{S}^\circ \cdot \boldsymbol{n}^\circ d\Sigma^\circ$ gives the amount of energy which flows into the vessel through $d\Sigma^\circ$ per unit time t°. Because the wall does no work in I°, this is just the flow of heat. Adding the contributions $\delta\boldsymbol{F}^{(v)}$ from all elements of the cylindrical wall at fixed time t, we get the total force from the cylindrical surface. Because the flow of heat is reversible, we may neglect the influençe of the synchronization and take the contributions $\boldsymbol{S}^\circ \cdot \boldsymbol{n}^\circ d\Sigma^\circ$ at fixed time t°. Thus we obtain from (41) Planck's force (36).

It is apparent from this analysis that, when calculating Planck's force, it is impossible to replace the fluid with the flow of heat by the fluid at rest with respect to the walls. Planck's force is caused just by the non-vanishing normal component of the velocity of the fluid at the wall of the standing vessel, together with the aberation which this velocity undergoes when passing to a system I, in which the vessel moves.

The total force on the fluid is the sum of the force (37) from the bottoms and Planck's force from the cylindrical surface. Using now the first law of thermodynamics, we get Planck's transformation formula for heat transferred to the fluid reversibly by conduction.

5. *Transformation of heat transferred to the system by a general process*

We explained the dynamical origin of Planck's force in the simple situation when heat flows reversibly into a perfect fluid through fixed diathermic walls. The heat transforms in this case according to Planck's formula (7). Let us now study the general situation when heat flows irreversibly into an arbitrary medium enclosed in rapidly deforming impenetrable walls.

We shall describe the medium by a symmetric energy-momentum tensor of unspecified structure. An element $d\Sigma$ of the moving impenetrable wall acts on the medium by the force (40). In the momentary rest system I° of an element in question, the element has the area $d\Sigma^\circ$ and normal $\boldsymbol{n}^\circ$. Let us define the four-vector $d\Sigma_i$ by the condition

$$d\Sigma^\circ_i = n^\circ_i d\Sigma^\circ, \; n^\circ_i = (\boldsymbol{n}^\circ, 0).$$

In an arbitrary system I, in which the element moves with velocity $\boldsymbol{v}$, the four-vector $d\Sigma_i$ has components

$$d\Sigma_i = \left(\frac{\boldsymbol{n}d\Sigma}{\sqrt{1-v^2}}, \; i\frac{\boldsymbol{n}\cdot\boldsymbol{v}d\Sigma}{\sqrt{1-v^2}}\right). \tag{42}$$

The vector $\boldsymbol{n}$ is now the normal to the element in I. The four-vector $d\Sigma_i$ is perpendicular to the four-velocity V_i of the element,

$$V_i d\Sigma_i = 0.$$

The force (40) can be expressed in the form

$$\delta F_\iota = T_{\iota k}\, d\Sigma_k \sqrt{1-v^2}. \tag{43}$$

The work $\delta A/dt$ done per unit time is

$$\frac{\delta A}{dt} = v_\iota \delta F_\iota = T_{\iota k} V_\iota d\Sigma_k (1-v^2). \tag{44}$$

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In the same way as the force δF_ι, we can determine the amount of energy which flows through the surface into the medium per unit time. If the surface is at rest the energy

$$S_\iota n_\iota d\Sigma = -iT_{4\iota}n_\iota d\Sigma$$

flows through it in the direction $\boldsymbol{n}$ per unit time. If the surface moves through medium in which the flow of energy vanishes, the energy

$$hv_\iota n_\iota d\Sigma = -T_{44}v_\iota n_\iota d\Sigma$$

escapes through it in the direction $-\boldsymbol{n}$. The energy $\delta H/dt$ which flows through the surface $d\Sigma$ in the direction $\boldsymbol{n}$ per unit time is therefore equal to

$$\frac{\delta H}{dt} = (-iT_{4\iota}+T_{44}v_\iota)n_\iota d\Sigma. \tag{45}$$

This expression can be written as

$$\frac{\delta H}{dt} = -T_{4i}V_4 d\Sigma_i(1-v^2). \tag{46}$$

Using now the first law of thermodynamics (4) for the definition of heat $\delta Q/dt$ which flows to the medium through the surface $d\Sigma$ per unit time, we get from (44) and (46)

$$\frac{\delta Q}{dt} = -T_{ik}V_i d\Sigma_k(1-v^2). \tag{47}$$

In the momentary rest system of the surface

$$\frac{\delta Q^\circ}{dt^\circ} = -T^\circ_{ik}V^\circ_i\, d\Sigma^\circ_k = -T_{ik}V_i d\Sigma_k. \tag{48}$$

Because $dt^\circ = dt\sqrt{1-v^2}$, the combination of equations (47), (48) gives just Planck's transformation law

$$\delta Q = \delta Q^\circ \sqrt{1-v^2} \tag{49}$$

from the heat δQ°, which flows through the element of the surface into the medium in the momentary rest system I° of the element, to the heat δQ, which flows through the element in I.

The formula (49) is quite general and holds for any reversible or irreversible process. It does not matter how rapidly the surface moves or how rapidly the heat flows. In this sense Planck's formula is universal. Its restrictions arise only when we try to apply it to heat transferred to the system through the whole surface and/or during the whole process. Indeed, the different elements of the wall move with different velocities and it is necessary to specify what system is taken for the global rest system of the thermodynamic system. Even then, it is not clear whether by putting together the elementary Planck's laws, relating to the transformation of elementary heats when passing from the local rest systems I° of elements $d\Sigma$ to the system I, we get the global Planck's law for the total heat, which relates to the transformation from the global rest system I^0 to I.

A typical process like this was described in the Introduction. There exists a system I^0, in which the thermodynamic system is enclosed up to the moment t_1^0 by adiabatic walls at rest. At t_1^0, some adiabatic walls are replaced by diathermic walls and the walls in general begin to move. The heat flows through the diathermic walls from the surroundings to the system and the moving walls do work. At t_2^0, the walls stop in the system I^0 and the thermodynamic system is again enclosed by adiabatic walls. The system I^0, in which the walls before and after the process are at rest, is called the global rest system of the thermodynamic system. If we wish, we may suppose that the thermodynamic system was in thermodynamic equilibrium up to the moment t_1^0. Also after the moment $t_2^0+\tau$, where τ is large in comparison with the time of relaxation of the thermodynamic system, the system is in thermodynamic equilibrium. We can now apply the first law of thermodynamics in the form (1) to the transition from the initial state at $t_{(i)}^0 < t_1^0$, to the final state at $t_{(f)}^0 > t_2^0+\tau$. When looking at the same transition from I, in which I^0 moves with the velocity $\boldsymbol{w}$, we choose the moments $t_{(i)}$ and $t_{(f)}$ so that the space-like hypersurfaces $t = t_{(i)}$ and $t = t_{(f)}$ intersect the walls at the moments when the walls are still or again at rest, and the elements of the thermodynamic system at the moments when the thermodynamic system is still or again in equilibrium.

The heat ΔQ transferred to the system during the transition from an initial state (i) to the final state (f) is obtained by integrating (47) at first over the surface Σ at time t and then over the time from $t_{(i)}$ to $t_{(f)}$,

$$\Delta Q = \int\limits_{t_{(i)}}^{t_{(f)}} dt \int\limits_{\Sigma(t)} -T_{ik}V_i d\Sigma_k(1-v^2).$$

The integration may be performed so that we integrate over the three-dimensional surface Ω representing the history of the wall, from the two-dimensional surface $\Sigma(t_{(i)})$, which is the intersection of Ω with the hyperplane $t = t_{(i)}$, to the two-dimensional surface $\Sigma(t_{(f)})$, which is the intersection of Ω with the hyperplane $t = t_{(f)}$,

$$\Delta Q = \int\limits_{\Sigma(t_{(i)})}^{\Sigma(t_{(f)})} -T_{ik}V_i n_k\sqrt{1-v^2}d\Omega. \tag{50}$$

The symbol $d\Omega$ denotes an invariant element $d\Sigma^\circ dt^\circ$ of Ω and n_i is the unit normal to Ω. Of course, the limits of integration $\Sigma(t_{(i)})$, $\Sigma(t_{(f)})$ can be deformed into the surfaces $\Sigma(t_{(i)}^0)$, $\Sigma(t_{(f)}^0)$, or into the surfaces $\Sigma(t_1^0)$, $\Sigma(t_2^0)$, because the integrand vanishes at the adiabatic walls.

Let us now analyse a number of special examples:

1) In I^0, the walls are at rest and heat flows (reversibly or irreversibly) to the system. In I, the walls move with a constant velocity and equation (50) gives

$$\Delta Q = \sqrt{1-w^2}\int\limits_{\Sigma(t_{(i)})}^{\Sigma(t_{(f)})} -T_{ik}V_i n_k d\Omega$$

$$= \sqrt{1-w^2}\int\limits_{\Sigma(t_{(i)}^0)}^{\Sigma(t_{(f)}^0)} -T^0_{ik}V^0_i n^0_k d\Omega = \Delta Q^0\sqrt{1-w^2}.$$

The heat transforms according to Planck's formula when passing from I^0 to I. The result of section 4 is thereby generalized for an arbitrary medium and an arbitrary shape of the container.

2) In I^0, the walls slowly move and heat flows through them to the system.

Neglecting the terms $(v^0)^2$ against unity, we put 1 for $\sqrt{1-(v^0)^2}$ and $\sqrt{1-w^2}$ for $\sqrt{1-v^2}$. The rest of the argument is the same as in the previous example. In this way, Planck's formula is proved for the case when the walls do work reversibly. Therefrom we get Planck's transformation formula for temperature (9) in the classical manner.

3) In I^0, the diathermic walls are at rest (or move slowly), while the adiabatic pistons can move rapidly and perform work irreversibly.

In the integral (50) the contribution over the adiabatic pistons vanishes and we again get Planck's transformation formula.

4) In I^0, the vessel with adiabatic bottom and adiabatic cylindrical walls is at rest. The vessel is closed by a diathermic piston which moves with an arbitrary velocity v, and through which heat flows into the vessel.

4*a*) We study the process in a system I, in which the vessel moves with the velocity $\boldsymbol{w}$ perpendicular to the axis of the cylinder. According to the addition law of the velocities:

$$\sqrt{1-v^2} = \sqrt{1-w^2}\,\sqrt{1-(v^0)^2}.$$

The equation (50) gives

$$\Delta Q = \sqrt{1-w^2}\int\limits_{\Sigma(t_1^0)}^{\Sigma(t_2^0)} -T^0_{ik}V^0_i n^0_k d\Omega = \Delta Q^0\sqrt{1-w^2}.$$

Planck's formula holds good even if heat flows through a wall which moves rapidly but transversally to the velocity $\boldsymbol{w}$.

4*b*) We study the process in a system I, in which the vessel moves with the velocity $\boldsymbol{w}$ parallel to the axis of the cylinder. According to the addition law of the velocities

$$\sqrt{1-v^2} = \sqrt{1-w^2}\,\sqrt{1-(v^0)^2}/(1\pm wv^0).$$

The equation (50) now gives

$$\Delta Q = \sqrt{1-w^2}\int\limits_{\Sigma(t_1^0)}^{\Sigma(t_2^0)} (1\pm wv^0)^{-1}(-T^0_{ik}V^0_i n^0_k\sqrt{1-(v^0)^2}d\Omega$$

and the heat ΔQ cannot be simply expressed as a function of the heat ΔQ^0.

Summarizing our results, we see that Planck's formula is valid for all reversible or irreversible processes, with the exception of those irreversible processes in which some heat is transferred to the system through the wall moving in I^0 rapidly in the same direction as I^0 moves with respect to I.

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6. Is transformation of heat a matter of definition of force?

Comparing the formulae (43), (44), and (47), we see that we can construct the four-vector

$$\delta F_i = T_{ik} d\Sigma_k = \left(\frac{\delta F_\iota}{\sqrt{1-v^2}},\ i\,\frac{\delta A/dt + \delta Q/dt}{\sqrt{1-v^2}} \right) \tag{51}$$

from the quantities δF_ι, δA, δQ; δF_i is the four-force from the elementary surface on the medium. The formula (51) makes it clear that the flow of heat does not supply the system with momentum, although it supplies it with energy. The change of momentum is entirely due to the mechanical force. This is connected with our definition of force as that part of the momentum transferred through the surface per unit time, which is not carried into the medium by penetrating particles, together with the fact that the walls are impenatrable.

However, the concept of force is not quite unambiguous. We introduce the concept of force in relativistic mechanics so that it should correspond as closely as possible with the concept of force in Newtonian mechanics. Of course, the relativistic mechanics does not coincide with Newtonian mechanics and the correspondence can never be perfect. For this reason, force can be identified with various physical quantities.

To illustrate this assertion, let us take an example from the dynamics of mass points with constant rest mass m_0. The equations of motion are

$$\frac{d}{d\tau}(m_0 V_i) = F_i,\ F_i V_i = 0;$$

τ is the proper time of the particle, V_i its four-velocity, and F_i the four-force. It is customary to split this equation into space and time components,

$$\frac{dm\boldsymbol{v}}{dt} = \boldsymbol{F}, \tag{52}$$

$$\frac{dm}{dt} = \boldsymbol{F}\cdot\boldsymbol{v}, \tag{53}$$

where

$$F_i = \left(\frac{\boldsymbol{F}}{\sqrt{1-v^2}},\ i\,\frac{\boldsymbol{F}\cdot\boldsymbol{v}}{\sqrt{1-v^2}} \right),\ m = \frac{m_0}{\sqrt{1-v^2}}, \tag{54}$$

and interpret the quantity $\boldsymbol{F}$ as force.

However, this identification is not unique. The equation (52) can be written in the form

$$m\frac{d\boldsymbol{v}}{dt} = \overline{\boldsymbol{F}} \equiv \boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{v})\boldsymbol{v}, \tag{55}$$

which is also closely analogous to the classical equation of motion. Therefore, one is tempted to identify $\overline{\boldsymbol{F}}$ with force.

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The reasons why this identification was never seriously considered are clear. At first, the prototype of dynamical theory in the early years of the special theory of relativity was Maxwell's electrodynamics. With the usual definition of force, the Lorentz force from electric field $\boldsymbol{E}$ and magnetic Field $\boldsymbol{H}$ on a moving charge e has the form

$$\boldsymbol{F} = e\,(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{H}),$$

well known from classical physics. On the other hand, for $\boldsymbol{F}$ we get the expression

$$\overline{\boldsymbol{F}} = e(\boldsymbol{E}-(\boldsymbol{E}\cdot\boldsymbol{v})\boldsymbol{v}+\boldsymbol{v}\times\boldsymbol{H}),$$

which differs from the classical one in terms of the second order in v. An advantage of the current identification of force with the quantity $\boldsymbol{F}$ is that when passing from classical electrodynamics to relativistic electrodynamics, only the left-hand side of the equation of motion changes (the mass of the particle depends on velocity), while the right-hand side (an expression for force) remains unchanged. That is why the equation (53) served only as an intermediary step to a further modification of the concept of mass (to the introduction of concepts of longitudinal and transversal mass).

The second reason why the identification of $\overline{\boldsymbol{F}}$ with force is not useful lies in the law of conservation of energy. Introducing $\overline{\boldsymbol{F}}$ into the equation (53), we get

$$\frac{dm}{dt} = \frac{\overline{\boldsymbol{F}}\cdot\boldsymbol{v}}{1-v^2}\,. \tag{56}$$

If we identify $\overline{\boldsymbol{F}}$ with force, the time derivative of energy is no longer equal to the work done by force per unit time. We should therefore admit that the law of conservation of energy must be modified at high velocities. When we define the force as usual, only the concept of energy is modified (the kinetic energy $\frac{1}{2}\,m_{\text{o}}\,v^2$ is replaced by the relativistic energy $m_{\text{o}}/\sqrt{1-v^2}$), but the classical form of the law of conservation of energy remains unchanged.

For both reasons it is apparent that the old definition of force $\boldsymbol{F}$ is more useful than the new definition of force $\overline{\boldsymbol{F}}$, because it preserves a closer analogy of relativistic physics with classical physics. We cannot, of course, proclaim that the new definition of force is uncorrect, but can prove it is disadvantageous.

We analyzed this example in order to clarify the definitional and factual nature of the concept of force, though it is not immediately connected with the controversy between Planck's and Ott's conceptions of heat. However, our second example is vital for this controversy.

If the particle moves under the influence of a four-force F_i which is not perpendicular to the four-velocity,

$$\frac{d}{d\tau}\,(m_{\text{o}}V_i) = F_i,\;\; F_iV_i \neq 0, \tag{57}$$

the rest mass of the particle changes during the motion,

$$\frac{dm_{\text{o}}}{d\tau} = -F_iV_i\,. \tag{58}$$

The equation (57) can be again split into the space and time components

$$\frac{dm\boldsymbol{v}}{dt} = \boldsymbol{F}, \tag{59}$$

$$\frac{dm}{dt} = \boldsymbol{F}\cdot\boldsymbol{v} + \frac{dQ}{dt}. \tag{60}$$

The four-force is now expressed by means of the quantities $\boldsymbol{F}$ and dQ/dt.

$$F_i = \left(\frac{\boldsymbol{F}}{\sqrt{1-v^2}},\ i\,\frac{\boldsymbol{F}\cdot\boldsymbol{v}+dQ/dt}{\sqrt{1-v^2}}\right). \tag{61}$$

According to Planck's ideas, it is usual to take the equations (59) and (60) for the equations of motion of a small particle to which heat is transferred. The quantity $\boldsymbol{F}$ is interpreted as the force acting on the particle and the quantity dQ/dt as the amount of heat transferred to the particle per unit time. The heat carries energy but no momentum to the particle; the change of momentum is due entirely to the mechanical force $\boldsymbol{F}$.

Because

$$-F_iV_i = \frac{dQ/dt}{1-v^2}$$

is a scalar, we get

$$\frac{dQ}{dt} = \frac{dQ^\circ}{dt^\circ}(1-v^2);$$

I° is the momentary rest system of the particle. The heat transferred to the particle in the time dt transforms therefore according to Planck's formula

$$dQ = dQ^\circ\sqrt{1-v^2}.$$

Ott's conception is based on a redefinition of force acting on the particle with vanishing rest mass. The four-force F_i is at first decomposed into the tangential and the normal directions to the world-line of the particle,

$$F_i = F_i^{\|}+F_i^{\perp},\quad F_i^{\perp}V_i = 0,$$

where

$$F_i^{\|} = -F_kV_kV_i,\quad F_i^{\perp} = F_i+F_kV_kV_i.$$

The normal component $F_i^{\perp}$ is then connected with the force $\boldsymbol{F}_O$ in the same manner as the four-force acting on the particle with constant rest mass is connected with the ordinary force $\boldsymbol{F}$,

$$F_i^{\perp} = \left(\frac{\boldsymbol{F}_O}{\sqrt{1-v^2}},\ i\,\frac{\boldsymbol{F}_O\cdot\boldsymbol{v}}{\sqrt{1-v^2}}\right). \tag{62}$$

The tangential component $F_i^{||}$ is called the heat four-vector and its components are identified in a similar manner as *e. g.* the components of the electric four-current,

$$F_i^{||} = \left(\frac{d\boldsymbol{Q}_O/dt}{\sqrt{1-v^2}},\ i\frac{dQ_O/dt}{\sqrt{1-v^2}} \right). \tag{63}$$

The quantity dQ_O/dt is interpreted as the amount of heat transferred to the particle per unit time and the quantity $d\boldsymbol{Q}_O/dt$ as the amount of momentum carried by heat to the particle per unit time. There is

$$F_i^{\perp} = \frac{dQ_O^\circ}{dt^\circ} V_i, \qquad \frac{dQ_O^\circ}{dt^\circ} = -F_k V_k.$$

We see that

$$d\boldsymbol{Q}_O = dQ_O\, \boldsymbol{v}$$

and

$$dQ_O = \frac{dQ_O^\circ}{\sqrt{1-v^2}}, \tag{64}$$

$$d\boldsymbol{Q}_O = \frac{dQ_O^\circ}{\sqrt{1-v^2}} \boldsymbol{v}. \tag{65}$$

The heat transferred to the particle manifests itself by (64), (65) as if a corpuscule of rest mass dQ_O° mowing with the same velocity as the particle was added to it, so that it carries to the particle the inertial mass (64) and the momentum (65) in the system I.

The equations of motion according to (62) and (63) are

$$\frac{dm\boldsymbol{v}}{dt} = \boldsymbol{F}_O + \frac{d\boldsymbol{Q}_O}{dt}, \tag{66}$$

$$\frac{dm}{dt} = \boldsymbol{F}_O \cdot \boldsymbol{v} + \frac{dQ_O}{dt}. \tag{67}$$

The momentum of the particle changes both because the external force $\boldsymbol{F}_O$ acts on the particle and because the heat carries to it the momentum $d\boldsymbol{Q}_O$. By virtue of (58),

$$dm^\circ = dQ_O^\circ$$

and Ott's force can be defined as that part of the change of momentum of the particle per unit time which is not associated with the supply of rest mass to the particle,

$$\boldsymbol{F}_O = \frac{dm\boldsymbol{v}}{dt} - \frac{dm^\circ}{dt}\frac{\boldsymbol{v}}{\sqrt{1-v^2}} = m^\circ \frac{d}{dt}\frac{\boldsymbol{v}}{\sqrt{1-v^2}}. \tag{68}$$

The heat transferred to the particle per unit time is, as well as in Planck's argument, that part of increase in the energy of the particle which is not due to the work of external force per unit time. The difference between Ott's and Planck's treatment of the equation of motion is thus entirely a matter of the new definition of force. Ott's interpretation of the equation of motion (68) is analogous to the interpretation of Mescerski's equation of motion for a classical particle with changing mass.

The equation (68) can be obtained also by projecting the equation (57) into the directions perpendicular and tangential to the world-line of the particle. We get the four-vector equation

$$m_0 \frac{dV_i}{d\tau} = F_i^{\perp} \tag{69}$$

and the scalar equation $dm_0/d\tau = -F_i^{||} V_i$. The space components of the equation (69) give just the equation (68), whle the time component is

$$\boldsymbol{F}_O \cdot \boldsymbol{v} = \frac{dm}{dt} - \frac{dm_0/dt}{\sqrt{1-v^2}} = m_0 \frac{d}{dt} \frac{1}{\sqrt{1-v^2}}. \tag{70}$$

By comparing the equations (61), (62) and (63), we get the relations between Ott's and Planck's quantities,

$$\boldsymbol{F} = \boldsymbol{F}_O + dQ_O \boldsymbol{v}, \qquad \boldsymbol{F}_O = \boldsymbol{F} - \frac{dQ}{1-v^2} \boldsymbol{v},$$

$$dQ = dQ_O(1-v^2), \qquad dQ_O = \frac{dQ}{1-v^2}. \tag{71}$$

From the known tansformation formula for Planck's heat we get the transformation formula for Ott's heat,

$$dQ_O = \frac{dQ}{1-v^2} = \frac{dQ_O}{\sqrt{1-v^2}}.$$

The transformation law for heat is therefore dependent on the identification of force.

It seems therefore that also for the system enclosed in impenetrable walls the transformation law of heat depends on the definition of force from walls on the system. Our results were obtained under the assumption that force is that part of momentum transferred through the wall to the medium which is not carried into the medium by penetrating particles. However, the four-force δF_i from the elementary surface on the medium can be treated in the same manner as the four-force F_i acting on a particle. We decompose the four-force (51) into the component perpendicular to the four-velocity of the surface, and the component parallel to the four-velocity of the surface,

$$\delta F_i = \delta F_i^{\perp} + \delta F_i^{||},$$

$$\delta F_i^{\perp} = (T_{ik} + T_{lk} V_l V_i)\, d\Sigma_k, \tag{72}$$

$$\delta F_i^{||} = -T_{lk} V_l d\Sigma_k V_i. \tag{73}$$

Ott's force $\delta \boldsymbol{F}_O$ form the surface on the medium is again introduced by the formula

$$\delta F_i^{\perp} = \left(\frac{\delta \boldsymbol{F}_O}{\sqrt{1-v^2}},\ i \frac{\delta \boldsymbol{F}_O \cdot \boldsymbol{v}}{\sqrt{1-v^2}} \right). \tag{74}$$

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The amount of heat $\delta Q_O/dt$ and the amount of momentum of heat $\delta \boldsymbol{Q}_O/dt$ transferred through the surface to the medium per unit time are obtained when identifying the components of $\delta F^{||}_i$,

$$\delta F^{||}_i = \left(\frac{\delta \boldsymbol{Q}_O/dt}{\sqrt{1-v^2}},\ i\,\frac{\delta Q_O/dt}{\sqrt{1-v^2}} \right). \tag{75}$$

By comparing the equations (43), (72) and (74), we get the relation between the old force $\delta \boldsymbol{F}$ and Ott's force $\delta \boldsymbol{F}_O$.

$$\delta \boldsymbol{F}_O = \delta \boldsymbol{F} - (-T_{ik} V_i d\Sigma_k)\, \boldsymbol{v}.$$

The term

$$-T_{ik} V_i d\Sigma_k = -T^\circ_{ik} V^\circ_i\, d\Sigma^\circ_k = S^\circ_\varkappa n^\circ_\varkappa d\Sigma^\circ$$

represents the energy $\delta H^\circ/dt^\circ$, which flows into the medium in the rest system of the elementary surface per unit proper time of the surface. It means that

$$\delta \boldsymbol{F}_O = \delta \boldsymbol{F} - \frac{\delta H^\circ/dt}{\sqrt{1-v^2}}\, \boldsymbol{v}.$$

Ott's force is therefore that part of the momentum transferred through the surface to the medium per unit time, which cannot be associated with the idea that in the rest system of the surface the energy δH° penetrates with an infinitesimal velocity, carrying the rest mass $\delta H^\circ/\sqrt{1-v^2}$ and the momentum $\delta H^\circ/\sqrt{1-v^2}\, \boldsymbol{v}$ in an arbitrary system I. Less exactly, Ott's force is that part of momentum transferred through the surface to the medium per unit time, which is not accompanied by a change of the rest mass.

In classical physics a surface which is impenetrable for particles is impenetrable for mass. In relativistic physics the particles of the medium can gain the energy (which manifests itself in an averaged macroscopical picture as the rest mass) by means of elastic collisions with the quick particles of the wall. Both definitions of force therefore differ.

For the heat transferred through the surface per unit time, we get by comparing (73) with (75)

$$\delta Q_O/dt = -T_{ik} V_i d\Sigma_k. \tag{76}$$

The quantity $\delta Q_O/dt$ is a scalar. For the heat δQ_O we get Ott's formula

$$\delta Q_O = \delta Q^\circ_O/\sqrt{1-v^2}.$$

The equation (76) coincides with the first law of thermodynamics, when the work of Ott's force is substituted for δA.

It seems therefore as if the transformation law of heat transferred throug a surface is a matter of convention how to define force. However, the situation is slightly different than for one particle with changing rest mass. Indeed, we possess a clear model of the mechanism which mediates the transfer of heat through the walls. No particles carrying rest mass with associated inertial mass and momentum flow into the medium. The flow of heat arises because the particles reflect from the wall with higher velocites than they had before, though they preserve their rest masses. For one particle, there is no difference between the current definition of force and Ott's definition of force, so that the force acting at the collision from the wall on the particle is uniquely defined. The time average of all microscopic forces by which an element of surface acts on the particles of medium is just the force $\delta \boldsymbol{F}$ and not Ott's force $\delta \boldsymbol{F}_O$. The force $\delta \boldsymbol{F}$ has therefore a real physical meaning, while Ott's force $\delta \boldsymbol{F}_O$ is a mere fiction.

The difference between Ott's force $\delta \boldsymbol{F}_O$ and the force $\delta \boldsymbol{F}$ arises in an averaged macroscopic picture from the fact that while the four-force acting on a particle of the medium is perpendicular to the four-velocity of the particle, the sum of such four-forces is not perpendicular to the four-velocity of the wall. To decompose the total four-force into the component perpendicular to the four-velocity of the wall and the component parallel to the four-velocity does not have a good physical meaning.

To illustrate our assertion that the force $\delta \boldsymbol{F}$ is a time average of actual forces by which the particles of wall act on the particles of the medium, we shall study a simple physical model of the diathermic wall. Let us imagine that a current of "black" particles of rest mass m_0 and the velocity a° falls on a current of "white" particles of the same mass m_0 and the velocity b°. The currents are synchronized so that the corresponding couples of particles collide always at the same place of the inertial system I°. Let us suppose for definiteness that $|a^\circ| > |b^\circ|$ and the velocities are almost antiparallel; a small angle between the velocities $\boldsymbol{a}^\circ$ and $\boldsymbol{b}^\circ$ is necessary only to ensure that the particles in the incoming and outgoing streams do not hinder each other's motion. The axis x° is oriented in the direction of a°.

According to the laws of elastic collision, the black and white particles simply exchange their velocities in I°. Because the black and white particles do not mix, we can take the place of their collision for an element of impenetrable wall. We denote by N° the number of collisions per unit time t°, *i. e.* N° is the number of particles which falls on the wall from each side.

The numerical density of particles with velocity a° (the number of particles per unit length in I°) is $\overset{\circ}{\varrho}_{(a)} = N^\circ/|a^\circ|$, and similarly $\overset{\circ}{\varrho}_{(b)} = N^\circ/|b^\circ|$. All other densities are also related to the unit of length or, in other words, to the unit of volume over the unit surface of the wall. We get

$$h^\circ = N^\circ \left(\frac{m_0}{|a^\circ|\sqrt{1-a^{\circ 2}}} + \frac{m_0}{|b^\circ|\sqrt{1-b^{\circ 2}}} \right),$$

$$g_1^\circ = S_1^\circ = N^\circ \left(\frac{m_0 a^\circ}{|a^\circ|\sqrt{1-a^{\circ 2}}} + \frac{m_0 b^\circ}{|b^\circ|\sqrt{1-b^{\circ 2}}} \right),$$

$$T_{11}^\circ = N^\circ \left(\frac{m_0 a^\circ}{\sqrt{1-a^{\circ 2}}} - \frac{m_0 b^\circ}{\sqrt{1-b^{\circ 2}}} \right) \tag{77}$$

for the white particles. The force which acts from the unit surface on the white particles is according to (40)

$$\delta F_1^\circ = T_{11}^\circ .$$

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The term in the brackets on the right-hand side of equation (77) represents the momentum gained by a white particle in a collisions, while N° is the number of collisions per unit time. Because $|a^\circ| > |b^\circ|$, $g_1^\circ = S_1^\circ > 0$, though the particle number current vanishes. This indicates that heat flows in the direction of higher velocity of the particles. Calculating the heat transferred through unit surface area of the wall from the black particles to the white particles explicitly according to (47), we get

$$\frac{dQ^\circ}{dt^\circ} = -T_{ik}^\circ V_i^\circ d\Sigma_k = S^\circ{}_1.$$

The flow of heat in I° is equal to the flow of energy. The white particles gain energy in collisions with the faster black particles and carry it further in the direction of x°.

We shall look at the process from the system I, in which I° moves with the velocity $w > 0$ in the direction of x°. The wall moves in I also with the velocity w and the number of collisions per unit time t will be

$$N = N^\circ \sqrt{1-w^2}.$$

The numerical density of white particles in the incomming stream will be

$$\varrho_{(b)} = N/(|b|+w),$$

because the particles move in I with the velocity $-w-|b|$ with respect to the wall Similarly

$$\varrho_{(a)} = N/|a-w|.$$

We can now easily determine the densities

$$h = N\left(\frac{m_0}{|a-w|\sqrt{1-a^2}} + \frac{m_0}{(|b|+w)\sqrt{1-b^2}}\right),$$

$$g_1 = S_1 = N\left(\frac{m_0 a}{|a-w|\sqrt{1-a^2}} + \frac{m_0 b}{(|b|+w)\sqrt{1-b^2}}\right),$$

$$T_{11} = N\left(\frac{m_0 a^2}{|a-w|\sqrt{1-a^2}} + \frac{m_0 b^2}{(|b|+w)\sqrt{1-b^2}}\right).$$

According to the equation (40)

$$\delta F_1 = N\left(\frac{m_0 a}{\sqrt{1-a^2}} - \frac{m_0 b}{\sqrt{1-b^2}}\right). \tag{78}$$

The expression in the brackets is the momentum gained by a white particle from a black particle in a collision. Because N means the number of collisions per unit time, δF_1 is just the time average of the forces by which the black particles act on the white particles across the wall.

Calculating the heat transferred through the wall per unit time by (47), we get

$$\frac{dQ}{dt} = N\left(\frac{m_0}{\sqrt{1-a^2}} - \frac{m_0}{\sqrt{1-b^2}}\right) -$$

$$-N\left(\frac{m_0 a}{\sqrt{1-a^2}} - \frac{m_0 b}{\sqrt{1-b^2}}\right) w.$$

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The first term represents the energy which the white particles gain from the black particles per unit time. This energy is not interpreted as heat by an observer in I. This observer sees that the wall (the place of contact between the black and white particles) macroscopically moves with the velocity w and concludes that the force (78) from the black particles must perform the macroscopic work on the white particles. He subtracts this work from the total energy which the white particles get from the black particles and he obtains the heat.

We investigated this simple model rather thoroughly in order to see that the force $\delta \boldsymbol{F}$ from the wall on the medium has a real meaning of the time average of the microscopic forces from the particles of the wall on the particles of the medium. We have seen at the same time how the collisions mediate the transfer of heat. Ott's examples of the transfer of heat are either based on inelastic collisions or on absorption of particles. Our example perhaps clarifies the fact that if heat is conducted through impenetrable wall, Planck's phenomenological approach corresponds to the physical nature of the phenomenon.

Accepting the natural definition of force, according to which the force from an impenetrable wall on the medium is equal to the flow of momentum into the medium, we get from the first law of thermodynamics an unambiguous result that for the heat transferred to the medium through an elementary surface Planck's transformation law holds. On the basis of the reversible process (2) we have investigated in Section 5 we must then conclude that Planck's formula for temperature is also valid.

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