STATIC MAXWELL-EINSTEIN-KLEIN-GORDON FIELDS

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This paper considers the existence of the stationary s state of matter field generating the most general statical metric and electrostatic fields. The system of equations are seen to be determinate, but if Weyl-Majumder condition between g_{44} and the electrostatic potential is assumed the system of equations becomes over determinate and it readily reduces to a determinate one if mass parameter equals the charge parameter in magnitude. Further, if the matter is at rest Weyl-Majumder condition is a necessary one and no solution exists unless the mass parameter in magnitude equals the charge parameter *i.e.*, matter cannot be at rest without the mass charge balance.

1. Introduction

Das [1] has introduced the Maxwell-Einstein-Klein-Gordon Field to have a complete singularity free model of matter. Das [1] and De [2], [3] obtained some solutions of the same set of field equations with different symmetries with mass charge balance. But, unfortunately, we get no elementary particle having such a balance of mass and charge. Here we shall be engaged in the general excistence of the static solutions and to see whether such a balance is necessary or not, at least when the metter is in equilibrium.

2. Notations and field equations

Units are so chosen $\hbar = c = G = 1$.

Latin indices stand for space-time components Greek indices for space only. An event and a space point will be denoted by x and $\boldsymbol{x}$ respectively, and time $t \equiv x^4$. The covariant differentiation of a tensor T: and partial differentiation of a scalar φ are denoted by

$$\nabla_k T: \text{ and } \varphi_a \equiv \frac{\partial \varphi}{\partial x^a}.$$

The combined Maxwell-Einstein-Klein-Gordon equations are taken to be [1] [2] [3] [4]:

$$B \equiv (D^i D_i + m^2)\psi = 0,$$

$$B^* \equiv (D^{*i} D_i^* + m^2)\psi^* = 0,$$

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$$E^i \equiv \nabla_j F^{ij} + \sqrt{4\pi}\, i\varepsilon(D^{*i}\psi^* \cdot \psi - \psi^* \cdot D^i\psi) = 0,$$

$$Q_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij}R + 8\pi[D^*_{hi}\, \psi^* \cdot D_{jh}\psi - g_{ij}(D^{*a}\psi^* \cdot D_a\psi - m^2\psi^*\psi) + E_{ij}] = 0,$$

where,

$$D_j \equiv \nabla_j + \sqrt{4\pi}\, i\varepsilon A_j,$$

$$D^*_j \equiv \nabla_j - \sqrt{4\pi}\, i\varepsilon A_j,$$

$$F_{ij} \equiv \nabla_j A_i - \nabla_i A_j,$$

$$D^*_{(i}\psi^* \cdot D_{j)}\psi \equiv D^*_i\psi^* \cdot D_j\psi + D^*_j\psi^* \cdot D_i\psi,$$

$$E_{ij} \equiv -F_{ij}F^K_j + \frac{1}{4} g_{ij}F_{ab}F^{ab},$$

$$\varepsilon \equiv (137)^{-1/2}$$

3. *Stationary s state of matter field*

Without loss of generality we can take for stationary s state of matter field:

$$\psi = u(\varkappa)\bar{e}^{iEt}, \qquad \psi^* = u(\varkappa)e^{iEt},$$

$$A_\alpha = 0, \qquad \varphi \equiv A_4(\varkappa) \neq 0, \tag{2}$$

and

$$ds^2 = -\bar{e}^{\omega(\varkappa)}[\bar{g}_{\alpha\beta}(\varkappa)d\varkappa^\alpha d\varkappa^\beta] + e^{\omega(\varkappa)}dt^2. \tag{3}$$

Therefore we get for $B = 0$,

$$-e^\omega \Delta_2 u + u\bar{e}^\omega\{m^2 e^\omega - [E - \sqrt{4\pi}\,\varepsilon\varphi]^2\} = 0, \tag{4}$$

$E^4 = 0$ becomes

$$-e^\omega(\bar{g})^{-1/2} \frac{\partial}{\partial \varkappa^\alpha} [(\bar{g})^{1/2}\bar{g}^{\alpha\beta}(\bar{e}^\omega \Phi_\beta)]$$

$$-2\sqrt{4\pi}\,\varepsilon\bar{e}^\omega \cdot u^2[E - \sqrt{4\pi}\,\varepsilon\varphi] = 0, \tag{5}$$

$$E^\alpha \equiv 0,$$

$\tilde{Q}_{\alpha\beta} = 0$ gives,

$$\bar{R}_{\alpha\beta} + \frac{1}{2}\omega_\alpha\omega_\beta - \frac{1}{2}\bar{g}_{\alpha\beta}\Delta_2\omega + 16\pi u_\alpha u_\beta + 8\pi m^2 e^{-\omega}\bar{g}_{\alpha 0}u^2 + \frac{1}{4} \cdot \frac{m^2}{\varepsilon^2}\bar{g}_{\alpha\beta}(\bar{g}^{\mu\nu}\omega_\mu\omega_\nu) -$$

$$-\frac{1}{2} \cdot \frac{m^2}{\varepsilon^2}\omega_\alpha\omega_\beta = 0, \tag{6}$$

$\tilde{Q}_{44} = 0$ becomes

$$e^{2\omega}\left[-\frac{1}{2}\Delta_2\omega + 8\pi m^2 e^{-\omega} \cdot u^2 + \frac{1}{4}\frac{m^2}{\varepsilon^2}\bar{g}^{\mu\nu}\omega_\mu\omega_\nu\right] = 0, \tag{7}$$

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and

$$\tilde{Q}_{\alpha 4} \equiv 0,$$

where

$$\tilde{Q}_{ij} \equiv Q_{ij} - \frac{1}{2} g_{ij} Q^{k}_{\cdot k}, \qquad \Delta_2 \omega \equiv \bar{g}^{\mu\nu} \omega_{||\mu\nu}$$

and horizontal bar above any quantity denotes the corresponding quantity with respect to $\bar{g}_{\alpha\beta}$ of the 3-space.

Now, we find that there are nine partial differential equations in (4) to (7) and there are exactly nine unknowns, namely u, φ, ω and six $\bar{g}_{\alpha\beta}$, that is, the system is a determinate one. Therefore, with proper Cauchy data Cauchy problem may be formulated. Hence the existence of the static stationary solutions without any mass charge balance is concluded.

Again let us see what happens when the Weyl–Majumder [5] [6] condition betwene g_{44} and the electrostatic potential, namely, $m^2 e^{\omega} = [E - \sqrt{4\pi}\, \varepsilon\varphi]^2$, is assumed. Here the equations (4) to (7) reduce to

$$\Delta_2 u = 0, \tag{8}$$

$$\Delta_2 \omega - \frac{1}{2} \bar{g}^{\alpha\beta} \omega_\alpha \omega_\beta - 16\pi \varepsilon^2 u^2 e^{-\omega} = 0, \tag{9}$$

$$\Delta_2 \omega - \frac{1}{2} \cdot \frac{m^2}{\varepsilon^2} \bar{g}^{\alpha\beta} \omega_\alpha \omega_\beta - 16\pi m^2 u^2 e^{-\omega} = 0, \tag{10}$$

$$\bar{R}_{\alpha\beta} + \frac{1}{2}\left(1 - \frac{m^2}{\varepsilon^2}\right) \omega_\alpha \omega_\beta + 16\pi u_\alpha u_\beta = 0. \tag{11}$$

The equations (8) to (11) contain again nine equations but the number of variables are now eight, *i.e.* we get an over determinate system of partial differential equations. But if $m^2 = \varepsilon^2$ the equations (9) and (10) are identical and we obtain again a determinate system.

From the above considerations we find that the Weyl–Majumder condition is not a necessary condition for the existence of the solutions of stationary s state of matter. Indeed, it is a restriction.

4. *Matter at rest*

For matter at rest generating the most general type of metric and electrostatic fields we can take (2) and (3) with $u = 1$, then $\Delta_2 u = 0$ and from (4) we get $m^2 e^{\omega} = [E - \sqrt{4\pi}\, \varepsilon\varphi]^2$. Therefore (9) and (10) become

$$\Delta_2 \omega - \frac{1}{2} \bar{g}^{\alpha\beta} \omega_\alpha \omega_\beta - 16\pi \varepsilon^2 e^{-\omega} = 0, \tag{12}$$

$$\Delta_2 \omega - \frac{1}{2} \cdot \frac{m^2}{\varepsilon^2} \bar{g}^{\alpha\beta} \omega_\alpha \omega_\beta - 16\pi m^2 e^{-\omega} = 0. \tag{13}$$

Subtracting (12) from (13)

$$\frac{1}{2}\left(1-\frac{m^2}{\varepsilon^2}\right)\bar{g}^{\alpha\beta}\omega_\alpha\omega_\beta+16\pi e^{-\omega}(\varepsilon^2-m^2)=0,$$

$$\bar{g}^{\alpha\beta}\omega_\alpha\omega_\beta=-32\pi\varepsilon^2e^{-\omega}, \tag{14}$$

if $m^2 \neq \varepsilon^2 (m^2 = \varepsilon^2$ case has been considered by Das [1]) Since $\bar{g}^{\alpha\beta}\omega_\alpha\omega_\beta$ is always positive, we can conclude that there exists no such solutions unless the positive definite character of $\bar{g}^{\alpha\beta}$ has been suspended. This suspension of the positive definite character of $\bar{g}^{\alpha\beta}$ means that the metric for the space-time manifol cannot be locally Minkowskian, therefore non-physical. However, if it allowed from (12) and (13) we get (14) and the following

$$\Delta_2\omega=0, \tag{15}$$

and now (11) reduce to

$$\bar{R}_{\alpha\beta}+\frac{1}{2}\left(1-\frac{m^2}{\varepsilon^2}\right)\omega_\alpha\omega_\beta=0 \tag{16}$$

If $\sqrt{1-\frac{m^2}{\varepsilon^2}}\cdot\omega=f$, (16), (15) and (14) reduce to

$$\bar{R}_{\alpha\beta}+\frac{1}{2}f_\alpha f_\beta=0, \tag{17}$$

$$\Delta_2 f=0 \tag{18}$$

and

$$\bar{g}^{\alpha\beta}f_\alpha f_\beta=-32\pi(\varepsilon^2-m^2)e^{-f\varepsilon/\sqrt{\varepsilon^2-m^2}} \tag{19}$$

The equations (17) and (18) are nothing but purely static gravitational field equations $R_{\alpha\beta}=0$, $R_{44}=0$, when $ds^2=-e^{-f(\vec{x})}.(\bar{g}(\vec{x})dx^\alpha dx^\beta)+e^{f(\vec{x})}\cdot dt^2$. We can construct the solutions of the combined field equations from purely gravitational cases provided the original metric of the gravitational field satisfies (19).

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