LOCAL QUANTUM FIELD THEORY WITHOUT DIVERGENCES

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The well known convergence difficulties within the framework of relativistic, local and microcausal quantum field theories may be avoided by introducing non-linear interactions represented by bounded operators. Such interactions mean infinite sets of many-body forces. The plural forces become more and more important for small distances and high concentrations of matter (strong fields). A general idea of how to perform approximative computations for such a theory is sketched.

All examples of the usual local and relativistic quantum field theory considered hitherto, with the Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}' \tag{1}$$

where $\mathcal{L}^{(0)}$ denotes a Lagrangian of a free field and $\mathcal{L}'$ denotes a Lagrangian of interaction, suffer unsurmountable convergence difficulties. In particular, one of the simplest examples of interaction: a self-interaction of a spinor field ψ given by a non-linear term

$$\mathcal{L}' = l^2(\bar{\psi}\psi)^2 \tag{2}$$

yields strongly divergent and non-renormalizable results. However, theoretical physicists — especially those working in the field of Axiomatic Field Theory — must have been blind by having overlooked the following very simple, in principle, possibility of a local quantum field theory with non-linear interaction:

The mathematical reason of the convergence difficulties is the fact that $\mathcal{L}'$ given by (2) or any other example of interaction considered hitherto, is an unbounded operator. Therefore, the convergence difficulties can be avoided by replacing the unbounded operator of interaction by a bounded operator. In particular, we may consider $\mathcal{L}'$ of the form

$$\mathcal{L}' = m\bar{\psi}\psi e^{-l^6(\bar{\psi}\psi)^2}. \tag{3}$$

This bounded operator may be developed into a power series

$$\mathcal{L}' = m\bar{\psi}\psi - ml^6(\bar{\psi}\psi)^3 + \dots \tag{3'}$$

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The first term in (3′) being nothing else but a mass-term, may be included into $\mathcal{L}^{(0)}$. The next term represents an interaction of the type of three-body forces whereas higher terms represent plural, $(2n+1)$-body forces. Another example of a bounded non-linear Lagrangian of interaction is

$$\mathcal{L}' = l^{-4}(1-e^{-l^{6}(\bar{\psi}\psi)^{2}}) \tag{4}$$

i.e.

$$\mathcal{L}' = l^{2}(\bar{\psi}\psi)^{2} - \frac{l^{8}}{2}(\bar{\psi}\psi)^{4} + \dots \tag{4'}$$

representing two-body and plural, $2n$ body forces.

The usual formalism of local quantum field theory applies without any modification to such cases of non-linear theories. The unitary scattering operator is of the usual form

$$U_{21} = Te^{i\int_{\sigma_1}^{\sigma_2} d^4x\mathcal{L}'} \tag{5}$$

where $\mathcal{L}'$ is given by (3) or (4), T is the operator of ordering in time while $\sigma_{1/2}$ are two space-like hypersurfaces of measurement. The usual procedure of taking normal products is also applicable.

Together with $\mathcal{L}'$ also the integral in (5) is a bounded operator provided the domain of integration is finite (the space integration is extended over a finite box of periodicity and σ_1 and σ_2 are not removed to $\pm$ infinity). In this case there are no reasons whatsoever for appearing of infinities. Such a theory is mathematically fully regular.

Most probably, the possibility of such a theory was never taken into the consideration for the following reason: it seems, at the first sight, that no practical method of computation exists if the interaction is of such a complicated form. Indeed, evaluation of the matrix elements of $\mathcal{L}'$ in a representation necessitates a power series expansion but we cannot break this power series after a finite number of terms because such breaking yields an unbounded operator so that we would run again into serious convergence difficulties, even more hopeless than in the simple case (2).

However, there exists a general method of dealing with this problem based on the following remark: breaking the power series (3′) or (4′) after a finite number of terms is meaningful if the field is sufficiently weak. This, in turn, is a condition upon the state of the field. E.g. we may define a state $|\rangle$ as representing a weak field if the following expectation value fulfils an inequality

$$-\langle|P_{\mu}^{(0)}P_{\mu}^{(0)}|\rangle < \frac{\alpha}{l} \tag{6}$$

where $P_{\mu}^{(0)}$ is the interaction-free part of energy-momentum of the field and α is a number. Therefore satisfactory results may be obtained by any method of approximations (perturbation calculus, Tamm–Dancoff method, *etc.*) provided we restrict ourselves to a limited class of state vector satisfying a condition of the form (6) and treat them as if they constituted a complete set. If only a limited number of terms in the expansion (3′) is taken into account

ACTA PHYSICA POLONICA Vol. XXXV (1969) Fasc. 3

and the rest is neglected the result of calculation cannot be improved by increasing the range (6) of the state vectors. On the contrary: by extending the range of state vectors beyond a certain limit we would get even a worse and worse approximation (tending to infinity). A better approximation may be obtained only by a simultaneous increase of both: the range of state vectors used for the computation and an appropriate increase of the number of terms in the expansion (3'). Thus, for practical computations we have to introduce a cutt off (in the space of state vectors). However, this cutt off does not mean an encroachment upon the formalism but is only a part of a logical and consistent method of approximations.

Our formalism incorporates a fundamental length which is unavoidable to secure the Lagrangian of interaction to be a bounded operator. It is also seen that any such Lagrangian consists in an infinite number of terms representing plural (many-body) interactions. This fact is not without interest for a tentative dynamics of quarks: in order to account for the facts that quark and antiquark (but not two quarks) form strongly bound systems (mesons) and three quarks form also strongly bound systems (baryons) an introduction of three-body (besides two-body) interactions seems unavoidable.

Let us add a remark about the problem of quantum electrodynamics. A replacement of the usual Lagrangian of interaction by a non-linear bounded operator, e.g.

$$\mathcal{L}' = j_\mu A_\mu \rightarrow j_\mu A_\mu e^{-l^8(j_\varrho A_\varrho)^2} \tag{7}$$

would spoil the gauge invariance of the formalism. The possibility that gauge invariance is violated for distances smaller than a certain fundamental length cannot be completely excluded, but if we wish to preserve strict gauge invariance, then the only possibility of a non-linear generalization consists in replacing the whole Lagrangian $\mathcal{L}$ (and not only its interaction part $\mathcal{L}'$) by a function of gauge invariant arguments

$$\mathcal{L} \rightarrow F(\mathcal{L}, f_{\mu\nu} f^{\mu\nu}) \tag{8}$$

representing a bounded operator. Together with $\mathcal{L}$ also any function of $\mathcal{L}$ is gauge invariant. This leads to a complicated nonlinear electrodynamics. However, its complications are rewarded by the fact that such electrodynamics is free of the usual convergence difficulties.

ACTA PHYSICA POLONICA Vol. XXXV (1969) Fasc. 3