ON THE ABELIAN FIELDS OF SPECIAL TYPE

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It is proved that the abelian Wightman field $A(x)$ is a polynomial in x variables. Some natural conditions are indicated under which this polynomial is a constant or vanishes.

1. Introduction

Let us consider a scalar neutral field which commutes with itself on the domain of its definition *i.e.*,

$$[A(x), A(y)] = 0 \quad \text{for every } x \text{ and } y. \tag{1.1}$$

We shall call such fields abelian.

Such a field need not vanish, *e.g.*, every translationally invariant operator satisfies (1.1). Of course, $A(x)$ safisfying (1.1) has to be reducible, unless the Hilbert space is one-dimensional. Moreover, in case the vacuum Ω is unique[1] it can not be cyclic with respect to $A(f)$, as the relation (1.1) together with the spectral condition implies also the vanishing of the two point Wightman function which in turn implies

$$A(f)\,\Omega = 0 \tag{1.2}$$

Since we are interested only in nonvanishing abelian fields — $A(x)$ can neither be local with respect to any field $B(x)$ whose cyclic vector is Ω nor have Ω as a cyclic vector for the commutant of the algebra of $A(x)$.

Araki [1] showed for local bounded operators that the elements of the center are translationally invariant. In case $A(f)$ under considerations is selfadjoint, it can be spectrally decomposed and for each spectral projection of this operator the theorem of Araki holds. However, Araki's proof makes an essential use of the boundedness of the operator and therefore it is unclear whether it can be extended to unbounded operators.

[1] Although the existence and uniqueness of the vacuum is irrelevant for the forthcoming considerations we may assume it to match physically interesting cases. We assume then also $(\Omega, A(f)\Omega) = 0$.

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It is rather obvious that the abelian Wightman fields are of little practical use in quantum field theory. They appear in some heuristic treatments of quantum fields in spaces with indefinite metric [2], which are out of the scope of our theorem presented below.

In this paper we consider the simplest abelian Wightman field satisfying some differential equation. The class of differential equations for which our considerations hold is large enough and includes as a special case the Klein-Gordon equation with finite mass term. We shall see that the case with vanishing mass term has to be singled out in our considerations.

We have shown that such an abelian field vanishes identically. There are several ways to prove this statement of which we are going to present two.

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2. Formulation of the Theorem

Let $A(f)$ be a scalar, neutral Wightman field [3] defined on D, which is linear and dense in $\mathcal{H}$. We assume that $f(x) \in \mathcal{S}$, *i.e.*, that $A(f)$ is tempered.

Let $A(f)$ be covariant with respect to a continuous representation $T(a)$ of the translations group

$$T(a)A(f)T(-a) = A(f_a) \tag{2.1}$$

where

$$f_a(x) = f(x-a)$$

$$T(a) = \int_{\bar{V}_+} \exp(iap)E(dp) \equiv \exp(i\hat{P}a) \tag{2.2}$$

$$\hat{P}a = \hat{P}_0 a_0 - \hat{\boldsymbol{P}}\boldsymbol{a}.$$

As usually we assume that $A(f)D \subset D$ and also $D(\Delta) = E(\Delta)D \subset D$ for any Borel set Δ in momentum space where

$$E(\Delta) = \int_{\Delta} E(dp), \tag{2.3}$$

Finally one assumes that $A(f)$ is abelian on D

$$[A(f), A(g)]D = 0 \quad \text{for all } f, g \in \mathcal{D} \tag{2.4}$$

and that it satisfies some differential equation of the form

$$(A, L^{+}(\partial)f)D = 0. \tag{2.5}$$

Such that the differential equation

$$L(\partial)Q(x) = 0 \tag{2.6}$$

does not admit any polynomial solution.

Our purpose is to prove the following

Theorem: The field $A(f)$ satisfying the conditions (1–6) vanishes identically on $\mathcal{H}$.

The proof of this Theorem consists of two steps. In the first step presented in two versions in Part 3 and 4 we prove that the field $A(x)$ is a polynomial in x. The socond, prese-in Part 5, is based on the lemma concerning the Klein–Gordon equation.

3. *The First Proof Showing that* $A(x)$ *is a polynomial in* x

We follow closely Araki's idea of proving the commutativity of the centrum of the algebra of local observables with the translations $T(a)$, [1].

Let us fix the Fourier transforms of $A(x)$ and $f(x)$ as follows

$$A(x) = (2\pi)^{-3/2}\int dp\tilde{A}(p)\exp(ipx)$$

$$f(x) = (2\pi)^{-3/2}\int dp\tilde{f}(p)\exp(-ipx). \tag{3.1}$$

In this case we have

$$A(f) = \int \tilde{A}(p)\tilde{f}(p)dp$$

$$[A(f)]^* = \int \tilde{A}(-p)\tilde{f}^*(p)dp. \tag{3.2}$$

With this convention of signs in exponents we have

$$A(f)D(\Delta) \subset D(\Delta + \operatorname{supp}\tilde{f})$$

$$[A(f)]^*D(\Delta) \subset D(\Delta - \operatorname{supp}\tilde{f}) \tag{3.3}$$

since

$$D(\Delta') \subset D(\Delta'') \quad \text{if} \quad \Delta' \subset \Delta''.$$

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Now for any set Δ_0 with the property

$$A(f)D(\Delta_0) = 0 \tag{3.4}$$

one may construct a sequence $\{\Delta_n\}$ of domains such that

$$A(f)D(\bigcup_{n=1}^{\infty}\Delta_n) = 0. \tag{3.5}$$

It is sufficient to assume only that

$$\Delta_n - \operatorname{supp}\tilde{f} \subset \Delta_{n-1}, \ (n = 1, 2, \ldots). \tag{3.6}$$

Indeed, we have then

$$[A(f)]^*A(f)D(\Delta_n) = A(f)[A(f)]^*D(\Delta_n) \subset A(f)D(\Delta_{n-1}). \tag{3.7}$$

Using induction method with respect to n we obtain

$$[A(f)]^*A(f)D(\Delta_n) = 0. \tag{3.8}$$

Taking into account the positive definiteness of the metric in $\mathcal{H}$ we obtain

$$A(f)D(\Delta_n) = 0, \ (n = 0, 1, \ldots). \tag{3.9}$$

Hence, using the fact that $D(\bigcup_{n=1}^{\infty} \Delta_n) \subset \bigcup_{n=1}^{\infty} D(\Delta_n)$, the result (3.5) follows.

Analogously, starting with the condition

$$[A(f)]^*D(\Delta_0') = 0 \tag{3.10}$$

and the chain of domains Δ_n' such that

$$\Delta_n' + \operatorname{supp} \tilde{f} \subset \Delta_{n-1}' \tag{3.11}$$

we obtain

$$[A(f)]^*D(\bigcup_{n=1}^{\infty} \Delta_n) = 0 \tag{3.12}$$

It is convenient to choose for Δ_0 and Δ_0' the complement of the closed upper cone

$$\Delta_0 = \Delta_0' = M \backslash \overline{V}_+ \tag{3.13}$$

Another possibility was indicated by Jancewicz [4]. Now, if the support of $\tilde{f}(p)$ contains points lying outside the $\overline{V}$, then the domains $\Delta_n \subset \Delta_{n-1} + \operatorname{supp} \tilde{f}$ will cover, in the limit $n \to \infty$, the whole upper cone $\overline{V}_+$. Therefore, we will have from (3.5)

$$A(f)D = 0 \tag{3.14}$$

since

$$E(M) = E(\overline{V}_+) = 1$$

Similarly, if support $\tilde{f}'(p)$ has points outside of $\overline{V}_+$ then $\Delta_n' \subset \Delta_{n-1}' - \operatorname{supp} \tilde{f}'$ will cover again the whole M. Hence, we will get

$$[A(f')]^*D = 0 \tag{3.15}$$

Comparing both relations we conclude that

$$(\psi_1, A(\varphi)\psi_2) = 0 \quad \text{for all} \quad \psi_1, \psi_2 \in D. \tag{3.16}$$

if $\tilde{\varphi}$ has the support outside of the point $p = 0$. Therefore, $\tilde{A}(p)$ is concentrated only in this point. As one knows [5] such a distribution is a finite sum of derivatives of the δ-function

$$(\psi_1, \tilde{A}(p)\psi_2) = \sum_{k=0}^{N} \tilde{C}_k(\psi_1, \psi_2)D^k\delta(p) \qquad N\text{-finite} \tag{3.17}$$

or in x-space,

$$(\psi_1, A(x)\psi_2) = \sum_{k=0}^{N} C_k(\psi_1, \psi_2)x^k$$

where

$$x^k = (x_0)^{k_0}(x_1)^{k_1}(x_2)^{k_2}(x_3)^{k_3}, \quad k = \sum_{i=0}^{3} k_i$$

Hence $A(x)$ is a polynomial in x.

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4. *The Second Proof Showing that $A(x)$ is a polynomial in x*

For the sake of simplicity we shall outline the proof giving up the mathematical rigour which, however, can be easily restored. We assume also for simplicity reasons, that the vacuum is unique (see the footnote on page 407).

For the 4-vector in the Minkowski momentum space $k \neq 0$ and $-k \in \overline{V}_+$ from the spectral condition follows

$$\tilde{A}(k)\Omega = 0 \tag{4.1}$$

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From (4.1) and (2.4) follows

$$[\tilde{A}(k)]^*\Omega = \tilde{A}(-k)\Omega = 0 \tag{4.2}$$

Eq. (4.1) holds for every $k \neq 0$. For $p-k \in \overline{V}_+$ and any vector-valued distribution satisfying

$$\hat{P}_\mu \psi(p) = p_\mu \psi(p), \qquad \mu = 0, 1, 2, 3$$

from the spectral condition as well as from (2.4) follows

$$\tilde{A}(k)\psi(p) = 0 \tag{4.3a}$$

$$[\tilde{A}(k)]^*\psi(p) = 0. \tag{4.3b}$$

From the conservation of energy and momentum (translational invariance) follows

$$(\tilde{A}(k)\psi(p), \psi(r)) = 0$$

for $k+r \neq p$.
But also for $p-2k \in \overline{V}_+$

$$(A(k)\psi(p), \psi(p-k)) = (\psi(p), [\tilde{A}(k)]^*\psi(p-k)) = 0$$

in virtue of (4.3). Thus again (4.3) holds for $p-2k \in \overline{V}_+$. An iteration procedure applied to $p-nk \in \overline{V}_+$ leads by induction to (4.3). Thus as long as $k \neq 0$ (4.3) remains valid since to every given $k \in \overline{V}_-$ and $p \in \overline{V}_+$ we are able to find sufficiantly large n so that $p-nk \in \overline{V}_+$. For $k \in \overline{V}_-$ we have

$$\tilde{A}(-k)\psi(p) = [\tilde{A}(k])^*\psi(p)$$

as well as

$$[\tilde{A}(-k)]^*\psi(p) = \tilde{A}(k)\psi(p)$$

both vanishing because $(-k) \in \overline{V}_-$.

The state Ω and the vector valued distributions $\psi(p)$ span the whole Hilbert space, thus

$$\tilde{A}(k) = [\tilde{A}(k)]^* = 0 \quad \text{for} \quad k \neq 0 \tag{4.4}$$

Since $\tilde{A}(k)$ is a distribution in the k-space with one-point support it is a polynomial in the x-space, *viz.*

$$A(x) = \sum_{k=0}^{N} c_k x^k$$

where C_k are operators defined on a dense set in $\mathcal{H}$, transforming under translations according to.

$$T(a)\, C_k T(-a) = \sum_{n=0}^{N} \binom{n}{k} C_n a^{n-k} \tag{4.5}$$

and commuting with one an other

5. *The Proof of the Theorem*

If $A(x)$ satisfies the equation $L(\partial)\, A(x) = 0$ which does not allow any polynomial solutions then $A(x)$-vanishes identically.

Clearly, when $A(x)$ is bounded the coefficients C_k, $k > 0$ vanish and $A(x) = A$ is a translationally invariant operator.

Now, we are going to verify that the Klein-Gordon operator $\Box - m^2$ with $m > 0$ is a good candidate for $L(\partial)$. Namely we shall prove the following:

Lemma. If $P_N(x)$ is any polynomial of N-th order satisfying the Klein-Gordon equation with non-zero mass then $P_N(x) = 0$.

Proof. The general form of $P_N(x)$ is the following

$$P_N(x) = \sum_{k=0}^{N} (\sum_{k_0,k_1,k_2,k_3} C_{k_0k_1k_2k_3} (x_0)^{k_0}(x_1)^{k_1}(x_2)^{k_2}(x_3)^{k_3}; k_0+k_1+k_2+k_3 = k \tag{3.18}$$

or symbolically

$$P_N(x) = \sum_{k=0}^{N} C_k x^k.$$

Acting with the operator $\Box - m^2$ on $P_N(x)$
we obtain the conditions on C_k

$$C_N = C_{N-1} = 0$$

$$m^2\, C_k = L(C_{k+2}) \tag{3.19}$$

where $L(C_{k+2})$ means the homogeneous linear combinations of coefficients of $k+2$ order.

Hence, all the coefficients C_k vanish when $m^2 \neq 0$. In the case $m = 0$ one can easily find a polynomial solutions in accordance with the d'Alembert solution of the wave equation.

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REFERENCES

[1] H. Araki, *Progr. Theor. Phys.*, **32**, 844 (1964).
[2] J. Lukierski, *Nuovo Cimento*, **54***A*, 583 (1968).
[3] R. F. Streater and A. S. Wightman, *PCT, spin and statistics and all that*, Benjamin, New York 1964.
[4] B. Jancewicz, *Acta Phys. Polon.*, in press.
[5] J. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. 2, Moscow 1958, (in Russian).