QUANTUM MECHANICS AS A QUANTUM MARKOVIAN PROCESS

By W. Garczyński

Institute of Theoretical Physics, University of Wrocław*

(Received November 10, 1968)

A new mathematical notion of the quantum Markovian process is introduced and it is shown that it is adequate for the needs of quantum mechanics.

We derived the basic content of the quantum theory starting from assumptions, which in simplest case of one spinless particle are the following:

If $(t, \boldsymbol{x}|s, \boldsymbol{y})$ denotes the probability amplitude of finding particle at the position $\boldsymbol{y}$ at the moment s when it is known that at $t < s$ the particle was at $\boldsymbol{x}$, then

(i) $$\int_{\mathcal{X}} d\boldsymbol{y}\, (t, \boldsymbol{x}|s, \boldsymbol{y})\, (s, \boldsymbol{y}|\tau, \boldsymbol{z}) = (t, \boldsymbol{x}|\tau, \boldsymbol{z})$$

(quantum causality condition),

(ii) $$\int_{\mathcal{X}} dy\, (t, \boldsymbol{x}|s, \boldsymbol{y})\, (s, \boldsymbol{y}|t, \boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{z})$$

unitarity condition),

(iii) $$\lim_{s\to t+0} \int_{\mathcal{X}} d\boldsymbol{x}\, \varphi(\boldsymbol{x})\, (t, \boldsymbol{x}|s, \boldsymbol{y}) = \varphi(\boldsymbol{y}) \text{ for any continuous } \varphi(\boldsymbol{x})$$

(continuity condition)

(iv) $$(t, \boldsymbol{x}|s, \boldsymbol{y}) = (s, \boldsymbol{y}|t, \boldsymbol{x})^*$$

(time reversal invariance)

(v) $$(t, \boldsymbol{x}|s, \boldsymbol{y}) = 0$$

on the boundary of $\mathcal{X}$, ($\mathcal{X}$ is the set of states attainable by a particle).

The motion described by the transition amplitude satisfying (i)—(iv) we call the quantum Markovian process. The word *quantum* serves as a distinction of this object from the classical

---

* Address: Instytut Fizyki Teoretycznej Uniwersytetu Wrocławskiego, Wrocław, Cybulskiego 36, Polska.

Markovian process in which the causality condition (i) is satisfied by the probability density instead of its amplitude.

The process is called the quantum Brownian motion if additionally there exists limits

$$A. \qquad \lim_{s\to t+0} (s-t)^{-1} \int_{\mathcal{X}} d\boldsymbol{y}(y_k-x_k)\,(t, \boldsymbol{x}|s, \boldsymbol{y}) = a_k(t, \boldsymbol{x})$$

$$B. \qquad \lim_{s\to t+0} (s-t)^{-1} \int_{\mathcal{X}} d\boldsymbol{y}(y_k-x_k)\,(y_j-x_j)\,(t, \boldsymbol{x}|s, \boldsymbol{y}) = b_{kj}(t, \boldsymbol{x})$$

$$C. \qquad \lim_{s\to t+0} (s-t)^{-1}\Big[\int_{\mathcal{X}} d\boldsymbol{y}(t, \boldsymbol{x}|s, \boldsymbol{y})-1\Big] = c(t, \boldsymbol{x})$$

$$D. \qquad \lim_{s\to t+0} (s-t)^{-1} \int_{\mathcal{X}} d\boldsymbol{y}(t, \boldsymbol{x}|s, \boldsymbol{y})\,(\boldsymbol{y}-\boldsymbol{x}, \nabla)^3\, \varphi(\boldsymbol{y}) = 0$$

for any continuous bounded function $\varphi(\boldsymbol{y})$.

Under these conditions, supplemented by the continuity and differentiability requirements, we have derived the two basic equations following the Kolmogorov method [1].

$$\left[\frac{\partial}{\partial t} - \frac{1}{2}\sum_{k_{ij}=1}^{3} \frac{\partial^2}{\partial x_k \partial x_j} b_{kj}(t, \boldsymbol{x}) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} a_k(t, \boldsymbol{x}) - c(t, \boldsymbol{x})\right] \psi(t, \boldsymbol{x}; \tau, u_0) = 0 \tag{1}$$

$$\left[\frac{\partial}{\partial \tau} + \frac{1}{2}\sum_{k_{ij}=1}^{3} b_{kj}(\tau, \boldsymbol{z}) \frac{\partial^2}{\partial z_k \partial z_j} + \sum_{k=1}^{3} a_k(\tau, \boldsymbol{z}) \frac{\partial}{\partial z_k} + c(\tau, \boldsymbol{z})\right] \varphi(\tau, \boldsymbol{z}; t, v_0) = 0 \tag{2}$$

where the wave functions are

$$\psi(t, \boldsymbol{x}; \tau, u_0) = \int_{\mathcal{X}} d\boldsymbol{z} u_0^*(\tau, \boldsymbol{z})\,(\tau, \boldsymbol{z}|t, \boldsymbol{x})$$

$$\psi(t, \boldsymbol{x}; \tau, u_0)\big|_{t=\tau} = u_0^*(\tau, \boldsymbol{x}) \tag{3}$$

$$\varphi(\tau, \boldsymbol{z}; t, v_0) = \int_{\mathcal{X}} d\boldsymbol{x}(\tau, \boldsymbol{z}|t, \boldsymbol{x})\, v_0(t, \boldsymbol{x}) = \psi^*(\tau, \boldsymbol{z}; t, v_0) \tag{4}$$

$$\varphi(\tau, \boldsymbol{z}; t, v_0)\big|_{t=\tau} = v_0(t, \boldsymbol{z}).$$

Furthermore, we have derived the following relations

$$-b_{kj}(t, \boldsymbol{x}) \equiv b_{kj}^*(t, \boldsymbol{x}),$$

$$a_k(t, \boldsymbol{x}) - a_k^*(t, \boldsymbol{x}) + \sum_{j=1}^{3} \frac{\partial b_{kj}^*(t, \boldsymbol{x})}{\partial x_j} \equiv 0, \tag{5}$$

$$c(t, \boldsymbol{x}) + c^*(t, \boldsymbol{x}) - \sum_{k=1}^{3} \frac{\partial a_k(t, \boldsymbol{x})}{\partial x_k} + \frac{1}{2}\sum_{k_j=1}^{3} \frac{\partial^2 b_{kj}(t, \boldsymbol{x})}{\partial x_k \partial x_j} \equiv 0, \tag{5}$$

which show that (1) and (2) are Schroedinger equations for the most general case. Indeed, if the process is stationary, homogeneous and isotropic then $(t, \boldsymbol{x}|s, \boldsymbol{y})$ is of the form

$$(t, \boldsymbol{x}|s, \boldsymbol{y}) = (s-t, (\boldsymbol{y}-\boldsymbol{x})^2), \tag{6}$$

ACTA PHYSICA POLONICA Vol. XXXV (1969) Fasc. 3

a_k, b_{kj}, c are constant and $b_{kj} = \delta_{kj}(i\beta)$, β—real, and we obtain

$$\left[i\hbar\frac{\partial}{\partial t}+\frac{\hbar\beta}{2}\Delta_x+i\hbar(\boldsymbol{a}, \nabla_x)-i\hbar c\right]\psi(t, \boldsymbol{x}; \tau, u_0) = 0,$$

$$\left[-i\hbar\frac{\partial}{\partial t}+\frac{\hbar\beta}{2}\Delta_z-i\hbar(\boldsymbol{a}, \nabla_z)-i\hbar c\right]\varphi(\tau, \boldsymbol{z}; t, v_0) = 0. \tag{7}$$

The identification of β with $\frac{\hbar}{m}$ yields the Schroedinger equations for ψ and ψ^* since a_k are real and c purely imaginary as it is clear from [5].

Thus we obtain the two Schroedinger equations, which generally differ from each other. The case with b_{ik} not constant and isotropic may be probably significant for solid state physics and quantum chemistry.

We have formulated such a theory also for the case of denumerable set of states, many body problem and relativistic quantum field theory, as well.

A systematic exposition of our results will be published soon. One sees that numerous attempts to incorporate quantum mechanics into the classical theory of the Markov processes (causality condition for probabilities instead of amplitudes) are not satisfactory (see e. g. [4] and other references given there). The potential $\frac{1}{i\hbar}c(t, \boldsymbol{x})$ vanishes identically for the probability density instead of an amplitude.

REFERENCES

[1] A. N. Kolmogorov, *Math. Ann.*, **104**, 415 (1931).
[2] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4-th ed., Clarendon Press, Oxford 1958.
[3] R. Feynman, *Rev. Mod. Phys.*, **20**, 367 (1948).
[4] L. de la Peña-Auerbach, E. Braun and L. S. Garcia Colin, *J. Math. Phys.*, **9**, 668 (1968).

ACTA PHYSICA POLONICA Vol. XXXV (1969) Fasc. 3