A GENERALIZATION OF CASE'S TRANSFORMATION OF THE DIRAC EQUATION

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A transformation of the inhomogeneous Dirac equation is proposed which: *1*) gives insight into the kinematical and dynamical aspects of the problem of separating positive and negative energy states of the Dirac particle, *2*) stresses the difference between the roles of the magnetic and electric stationary fields in such a procedure. The explicit formula for the transformed Hamiltonian is derived and interpreted. It is given in a finite form as well as in an expansion into a series. A simple geometrical interpretation follows the calculation.

I. Introduction

Nearly all problems dealing with the one-particle description of fermions, in particular those based on the scheme of Foldy and Wouthuysen [1] and on its futher generalizations are discussed separately for free particles and for particles in an external field, as rather different methods of approach are required in each case. The first case which, owing to its simplicity, can be treated without using any approximation became a starting point of many general investigations concerning *e. g.* the mathematical structure of the respective transformations, the formal definitions of some new one-particle observables and their subsequent physical interpretation[1]. On the other hand, the problem of interacting particles is less suitable for such general considerations, since, even if reduced to the simplified model of a single particle in an external field, it encounters many well-known difficulties directly related to its approximate character. The main attempt is therefore directed towards transforming the Hamiltonian into an, at least approximately "even" form (separating the positive and negative energy states). The most typical methods of achieving this purpose are those using expansions in powers of a suitable small parameter[2]. The convergence of

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[1] For details and references see *e. g.* Hanus *et al.* [2], [3].

[2] A detailed classification and analysis of such methods is given by Eriksen [4], [5] and Eriksen and Kolsrud [6].

such expansions is a rather delicate question and their consistent use requires, by turns, some restrictions imposed on the initial conditions for the particle as well as on the space-time variation of the applied field. Another important problem discussed by many authors is that of the uniqueness of the respective transformations[3]. A special case, intermediary to some extent, between the two previously mentioned (of a strict transformation for the free particle, or of an iterative procedure in the presence of external fields, respectively) is the transformation introduced by Case [10], for the electron in a stationary magnetic field (but with no electric field present). This transformation giving a strict separation of particle- and anti-particle states appears to be a simple generalization of that for the free particle. The possibility of such a generalization relies upon the fact that the stationary magnetic field does no work, thus causes no transitions between the states of opposite signs of energy, in contrast to the effect of an electric field.

There is no obstacle against a formal application of the same transformation (depending merely on the magnetic field) to the more general Dirac Hamiltonian containing contributions due to both, the magnetic and electric fields. Although no strict separation of the positive and negative energy states may be expected in this case, the mathematical structure of the transformed Hamiltonian and its further interpretation may be interesting for many reasons. These problems will be discussed in this paper. In the next chapter we derive the formulae for this generalized case transformation of the Dirac equation. The interpretation and a simple geometrical representation will be discussed in Chapter 3.

2. *The Case transformation in the presence of magnetic and electric stationary fields*

We start from the standard form of the Dirac Hamiltonian

$$H = \varrho_3 mc^2 + \varrho_1 c\boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) + eA_0. \tag{1}$$

A time-independent electromagnetic field expressed by the potentials $\boldsymbol{A}(\boldsymbol{x})$, $A_0(\boldsymbol{x})$ will be assumed throughout this paper. The use of the Dirac operators $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$, rather than $\boldsymbol{\alpha}$, β, will be convenient for our further calculations dealing mainly with the operator relations in $\boldsymbol{\rho}$-space. We thus follow the notation chosen by Tani [11]. Introducing the abbreviations

$$\boldsymbol{\pi} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}, \tag{2}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\pi} = \xi = mcz, \tag{3}$$

we have

$$H = mc^2(\varrho_3 + \varrho_1 z) + eA_0. \tag{4}$$

[3] Previously quoted papers [4]—[6] may be mentioned. Recently, this problem was systematically investigated by de Vries and Jonker [7], [8], [9]. These authors have shown the mutual equivalence of two commonly used iterative methods of successive elimination of the odd terms from the Hamiltonian; they established the precise conditions for the uniqueness of this procedure and performed explicit calculations to the 12^{th} order of approximation. A wide list of references on this subject may be found in the quoted papers of these authors.

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In accordance with (1)–(4), the unitary operator of Case (transforming (4) to the even form under the assumption $\boldsymbol{A} \neq 0$, $A_0 = 0$) may be expressed in the form

$$U = e^{iS}, \quad S = \frac{1}{2} \varrho_2 \varphi(z), \quad \varphi(z) = \text{arc tg } z, \tag{5}$$

or explicitly

$$U = U_1 + i\varrho_2 U_2, \quad U^\dagger = U_1 - i\varrho_2 U_2, \tag{6}$$

with

$$U_1 = \cos\frac{\varphi}{2} = \sqrt{\frac{E+mc^2}{2E}}, \quad U_2 = \sin\frac{\varphi}{2} = \sqrt{\frac{E-mc^2}{2E}}, \tag{7}$$

$$E = c\sqrt{m^2c^2+\xi^2} = mc^2\sqrt{1+z^2}. \tag{8}$$

We now apply the same transformation to the more general case, when both $\boldsymbol{A} \neq 0$ and $A_0 \neq 0$. Simple calculations give

$$H_U = UHU^\dagger = T_U + V_U^{(e)} + V_U^{(0)}, \tag{9}$$

where

$$T_U = \varrho_3 E, \tag{10}$$

$$V_U^{(e)} = e(U_1 A_0 U_1 + U_2 A_0 U_2), \tag{11}$$

$$V_U^{(0)} = i\varrho_2 e(U_2 A_0 U_1 - U_1 A_0 U_2). \tag{12}$$

We see that the presence of the electric field $\boldsymbol{E} = -\nabla A_0$ adds two further terms (11) and (12) to the transformed Hamiltonian. The latter term is odd.

An approximate expression for H_U, perhaps more suitable for explicit calculations than the strict formula (9), can be obtained in the non-relativistic limit (*i.e.* under the assumption $|z| \ll 1$), by expanding E and U_1, U_2 in powers of z. We have, namely,

$$E = mc^2\left(1+\frac{1}{2}z^2-\frac{1}{8}z^4+\frac{1}{16}z^6-\frac{5}{128}z^8+\dots\right)$$

$$= mc^2+\frac{1}{2m}\xi^2-\frac{1}{8m^3c^2}\xi^4+\frac{1}{16m^5c^4}\xi^6-\frac{5}{128m^7c^6}\xi^8\dots \tag{13}$$

$$U_1 = 1-\frac{1}{8}z^2+\frac{11}{128}z^4-\frac{69}{1024}z^6+\dots$$

$$= 1-\frac{1}{8m^2c^2}\xi^2+\frac{11}{128m^4c^4}\xi^4-\frac{69}{1024m^6c^6}\xi^6+\dots \tag{14}$$

$$U_2 = \frac{1}{2}z\left(1-\frac{3}{8}z^2+\frac{31}{128}z^4-\frac{187}{1024}z^6+\dots\right)$$

$$= \frac{1}{2mc}\xi\left(1-\frac{3}{8m^2c^2}\xi^2+\frac{31}{128m^4c^4}\xi^4-\frac{187}{1024m^6c^6}\xi^6+\dots\right), \tag{15}$$

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whence

$$T_U = \varrho_3 \left(mc^2 + \frac{1}{2m}\, \xi^2 - \frac{1}{8\, m^3c^2}\, \xi^4 + \frac{1}{16\, m^5c^4}\, \xi^6 - \frac{5}{128\, m^7c^6}\, \xi^8 + \dots \right), \tag{16}$$

$$V_U^{(e)} = e \Bigg[A_0 - \frac{1}{8\, m^2c^2} (\xi^2 A_0 - 2\xi A_0 \xi + A_0 \xi^2) +$$

$$+ \frac{1}{128\, m^4c^4} (11\xi^4 A_0 - 12\xi^3 A_0 \xi + 2\xi^2 A_0 \xi^2 - 12\xi A_0 \xi^3 + 11 A_0 \xi^4) -$$

$$- \frac{1}{1024\, m^6c^6} (69\xi^6 A_0 - 62\xi^5 A_0 \xi + 11\xi^4 A_0 \xi^2 - 36\xi^3 A_0 \xi^3 +$$

$$+ 11\xi^2 A_0 \xi^4 - 62\xi A_0 \xi^5 + 69 A_0 \xi^6) \dots \Bigg], \tag{17}$$

$$V_U^{(0)} = \frac{1}{2}\, ei\varrho_2 \Bigg\{ \frac{1}{mc}\, [\xi, A_0] - \frac{1}{8\, m^3c^3} (3\xi^3 A_0 - \xi^2 A_0 \xi + \xi A_0 \xi^2 - 3 A_0 \xi^3) +$$

$$+ \frac{1}{128\, m^5c^5} (31\xi^5 A_0 - 11\xi^4 A_0 \xi + 6\xi^3 A_0 \xi^2 - 6\xi^2 A_0 \xi^3 + 11\xi A_0 \xi^4 - 31 A_0 \xi^5) -$$

$$- \frac{1}{1024\, m^7c^7} (187\xi^7 A_0 - 69\xi^6 A_0 \xi + 31\xi^5 A_0 \xi^2 - 33\xi^4 A_0 \xi^3 +$$

$$+ 33\xi^3 A_0 \xi^4 - 31\xi^2 A_0 \xi^5 + 69\xi A_0 \xi^6 - 187 A_0 \xi^7) \dots \Bigg\}. \tag{18}$$

Introducing (16)–(18) into (9) and assuming that $\boldsymbol{p}$ now stands for $-i\hbar\nabla$, we obtain H_U expanded in a power series. The respective wave equation in configuration space can easily be written to an arbitrary order of approximation. In accordance with (2) and (3) we have

$$\xi = \boldsymbol{\sigma} \left(\boldsymbol{p} - \frac{e}{c}\, \boldsymbol{A} \right), \quad \xi^2 = \left(\boldsymbol{p} - \frac{e}{c}\, \boldsymbol{A} \right)^2 - \frac{e\hbar}{c}\, \boldsymbol{\sigma} \cdot \boldsymbol{H}, \quad \boldsymbol{H} = \mathrm{rot}\, \boldsymbol{A}, \tag{19}$$

$$[\xi, A_0] = i\hbar \boldsymbol{\sigma} \cdot \boldsymbol{E}. \tag{20}$$

Hence, $V_U^{(0)}$ can be also expressed in the form

$$V_U^{(0)} = -\frac{e\hbar}{2mc}\, \varrho_2 \Bigg\{ \boldsymbol{\sigma} \cdot \boldsymbol{E} - \frac{1}{8\, m^2c^2} [3\{\xi^2, \boldsymbol{\sigma} \cdot \boldsymbol{E}\} + 2\xi(\boldsymbol{\sigma} \cdot \boldsymbol{E})\xi] +$$

$$+ \frac{1}{128\, m^4c^4} [31\{\xi^4, \boldsymbol{\sigma} \cdot \boldsymbol{E}\} + 20\xi\{\xi^2, \boldsymbol{\sigma} \cdot \boldsymbol{E}\}\xi + 26\xi^2(\boldsymbol{\sigma} \cdot \boldsymbol{E})\xi^2] -$$

$$- \frac{1}{1024\, m^6c^6} [187\{\xi^6, \boldsymbol{\sigma} \cdot \boldsymbol{E}\} + 118\xi\{\xi^4, \boldsymbol{\sigma} \cdot \boldsymbol{E}\}\xi + 149\xi^2\{\xi^2, \boldsymbol{\sigma} \cdot \boldsymbol{E}\}\xi^2 + 116\xi^3(\boldsymbol{\sigma} \cdot \boldsymbol{E})\xi^3] \Bigg\}. \tag{21}$$

The electric field thus appears in (21) in terms of the expression

$$-\frac{e\hbar}{2mc}\, \boldsymbol{\sigma} \cdot \boldsymbol{E}$$

and of its derivatives, while the presence of the magnetic field is exhibited by the symbol ξ.

3. *Discussion of the results*

The general meaning of the transformation (5) (relating the Hamiltonians (1) and (9)) should be explained at first, as, except for the special case $A_0 = 0$, the transformed Hamiltonian (9) does not become even. The essentially different roles in which odd terms appear in these two Hamiltonians, respectively, may easily be discovered by comparing their operator properties in $\boldsymbol{\rho}$-space. The well-known characteristic feature of the Dirac Hamiltonian (1) is that its kinetic energy is a linear combination of both even and odd terms proportional to ϱ_3 and ϱ_1, respectively, while its potential energy is a c-number in this space. On the other hand, the transformation (5) changes the kinetic energy into an even operator (10) containing merely ϱ_3, but the odd term (12) containing ϱ_2 appears then in the potential energy, beside the other one (11), free of the $\boldsymbol{\rho}$-operators. Hence, we may say that term (12) expresses the dynamical effect of mixing the positive and negative energy states under the influence of the electric field, while all kinematical aspects leading to the most far reaching separation of these states have been taken into account by transformation (5).

It should be possible, in principle, to treat transformation (5) as a first step towards obtaining the even Hamiltonian in successive orders of approximation (under conditions which make this procedure consistent). We should thus combine the strict formula (5) with an iterative procedure eliminating the odd terms "of dynamical character" which have their source in the term (12) of the Hamiltonian (9). We do not follow these rather tedious calculations, as the resulting Hamiltonian is explicitly known, to a very high order of approximation, owing to the calculations made in different way by de Vries and Jonker ([7], [8], [9]). We prefer to restrict our present considerations to the Hamiltonian (9) and to the picture of the Dirac equation defined by transformation (5). The simple structure of the expansion of (5) into a series given by (16)–(18) allows to compare the ordering of its particular terms (10), (11) and (12) according to the powers of various parameters, such as e, m^{-1} or c^{-1}, alternatively used, depending on the circumstances. The ordering in powers of m^{-1} is more simple than that in powers of c^{-1} (*e.g.* of 137^{-1}, in the atomic unit system); therefore the latter is rather avoided in formal calculations, although it corresponds more closely to the physical classification of the respective terms regarding their order of magnitude (the Sommerfeld correction may be quoted as an example). Essentially, the expansion of the Hamiltonian (9) is that in powers of the dimensionless parameter p/mc, small in the non-relativistic limit.

The structure of the Hamiltonian (9) implies a simple geometrical interpretation of transformation (5), similar to that proposed by Saavedra [12] (see also Case [10] and Bjorken, Drell [13]) for the free particle[4]. Our notation using the Dirac operators $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$ instead of $\boldsymbol{\alpha}$, β is very useful for this purpose, as merely operator relations in $\boldsymbol{\rho}$-space are involved. For the free particle as well as in the presence of the magnetic field the Dirac Hamiltonian may be treated as a "vector" in the (ϱ_3, ϱ_1)-plane, its components being, moreover, dependent on other operators and on the parameters m, e and c. A transformation to the even form is equivalent to a "rotation" in this plane (*e.g.* about the perpendicular ϱ_2-axis) causing the

[4] Note, however, that the roles of fixed and rotated, perpendicular axes in the defined plane of rotation have been interchanged in our interpretation, as compared to that of Saavedra.

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rotated Hamiltonian to become "parallel" to the ϱ_3-axis. The angle of rotation $\frac{1}{2}\varphi = \frac{1}{2}\,\mathrm{arc\,tg}\,z$ (z being given by (3)) can be unambiguously defined, as the initial components of H in the directions ϱ_3 and ϱ_1 commute with each other. Substantial comlications are brought into this geometrical picture by the electric field, independently of the presence or absence of the magnetic field. Indeed, just the term eA_0 destroys the possibility of treating the Dirac Hamiltonian as a symbolic vector in ϱ-space and its transformation as a rotation in this space. Only the kinetic energy can be brought to coincide with the ϱ_3-axis, but no similar geometrical interpretation exists for the transformation, neither of the total Hamiltonian nor of the potential energy separately (as its two terms $V_U^{(e)}$, $V_U^{(0)}$, proportional to I and ϱ_2 respectively do not commute with each other).

More detailed considerations concerning the properties of the new picture of the Dirac equation obtained by means of the transformation (5) will be given separately.

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