A REVISED THEORY OF GENERAL RELATIVITY AND QUANTIZATION OF THE GRAVITATIONAL FIELD II

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By modifying the Lagrangian and introducing some coordinate conditions the theory of gravitation can be modified (without spoiling its most beautiful feature: the connection between gravitation and geometry). In contradistinction to the traditional theory there exist solutions representing truly localized wave packets, energy-momentum is localized, and a linear approximation is meaningful. The modified theory is quantizable.

1. *Modified field equations*

In Part I the foundations of the Theory of General Relativity, especially the principles of General Covariance and General Relativity, were criticised. It was pointed out that these principles are ambiguous and, if interpreted in Einstein's sense, can be regarded only as heuristic principles. It cannot be excluded that some geometrically distinguished coordinate systems are also physically distinguished and play an essential role in the problem of gravitation.

In order to construct a more satisfactory theory of gravitation we abandon the Principle of General Covariance (in its strong Einsteinnian sense) but retain the idea that gravitation is of geometrical origin, i.e. is expressible in terms of the tensor of curvature. Instead of the Postulate of General Covariance we assume that the harmonic (de Donder–Fock) coordinates satisfying the conditions

$$\Gamma^{\mu} = 0 \tag{1}$$

or, equivalently

$$(\sqrt{-g}\, g^{\varrho\lambda})_{,\varrho} = 0 \tag{1'}$$

are physically distinguished and play the role of a substitute of the inertial-cartesian coordinates of Special Relativity. Moreover, we abandon the traditional Lagrangian (equivalent to the scalar curvature R) and look for a new one satisfying the following requirements: it

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has to be bounded[1] and to yield second order field equations free of any singularities. Of course, the new Lagrangian should correspond to the traditional one except for the case of extremly strong fields.

An example of a Lagrangian satisfying the above requirements is

$$\mathcal{L} = \Lambda e^{-l^4\Lambda^2} \tag{2}$$

where Λ is the Lagrangian of the traditional theory

$$\Lambda = -g^{\mu\nu}\Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\varrho} \tag{3}$$

if expressed in harmonic coordinates and l is a fundamental length.

The field equations are derivable from the variational principle

$$\delta W = 0 \tag{4}$$

where

$$W = \int d^4x(\sqrt{-g}\mathcal{L} + \Phi_\lambda(\sqrt{-g}\, g^{\varrho\lambda})_{,\varrho}) \tag{5}$$

with $\mathcal{L}$ given by (2) and Φ_λ denoting the Lagrangian multipliers. Varying Φ_λ we obtain the coordinate conditions (1) and varying the metric tensor components we get further ten equations. Altogether, there are fourteen equations for the fourteen functions $g_{\mu\nu}$ and Φ_λ.

Inasmuch as for $l^4\Lambda^2 \ll 1$ our Lagrangian reduces to the traditional one, equivalent to the scalar curvature, the Einsteinnian Principle of General Covariance holds true for weak and moderately strong fields and breaks down only in the case of extremly strong fields[2]. There must exist also within the framework of our formalism, a solution of the Schwarzschild type for distances sufficiently far from the source but free of the Schwarzschild singularity.

Of course, our formalism remains generally covariant in the weak sense of this word: knowing that the quantities $g_{\mu\nu}$ transform like covariant tensor components under general coordinate transformations we may readily transcribe any solution of our field equations to arbitrary coordinates.

On the other hand, the existence of a preferred class of coordinates yields a smooth transition to the case of Special Relativity. Inasmuch as the Γ-symbols behave like components of a tensor under the group of all linear transformations, our action integral is invariant under the 20-parametric group of transformations

$$\bar{x}^\mu = a^\mu_\nu x^\nu + b^\mu \tag{6}$$

which secures the existence of twenty conservation laws.

If a class of coordinates is physically distinguished then also the generators of infinitesimal transformations of these coordinates are physically distinguished. According to Noether's theorem the invariance under the linear group of transformations (6) secures the existence of twenty conservation laws. Ten of them (connected with translations and rigid

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[1] Compare J. Rayski, *Acta Phys. Polon.* **35**, 403 (1969).

[2] Or if we introduce coordinates differing appreciably from harmonic ones for microscopic distances of the order l.

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rotations of the privileged coordinates) are interpretable as the laws of conservation of energy-momentum, angular momentum and velocity of the mass centre, as usually. The remaining ten conservation laws are connected with the invariance of the formalism under dillatations and distortions (because a^{μ}_{ν} appearing in (6) do not need to satisfy orthogonality conditions).

The most peculiar features of the traditional theory are directly connected with the fact that in the absence of sources G^{00} vanishes. The fact that the component τ^0_0 is expressible in the form of a 3-divergence follows directly from the tenth equation of Einstein $G^{00} = 0$. It is also responsible for the fact that the linear approximation is false, *i.e.* does not constitute an approximation to any exact solution of Einstein's theory[3]. In the modified theory G^{00} may differ from zero even in the absence of the sources of the gravitational field. Consequently, τ^0_0 is not of the form of a 3-divergence, and a linear approximation becomes meaningful. In the next section we shall discuss the linear approximation in more detail.

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2. *Linear approximation*

In the linear approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7}$$

where $h_{\mu\nu}$ are small, the Lagrangian (1) simplifies to

$$\mathcal{L} = \frac{1}{4}(h_{\mu\varrho,\sigma})^2 - \frac{1}{2}(h_{\mu\varrho,\varrho})^2 + \frac{1}{2} h_{\mu\varrho,\mu} h_{\sigma\sigma,\varrho} - \frac{1}{4}(h_{\mu\mu,\varrho})^2 \tag{8}$$

where we went over to the notation

$$x^0 = x_0,\; x^k = ix_k. \tag{9}$$

The linearized field equations derivable from (8) are

$$h_{\mu\nu,\varrho\varrho} - h_{\mu\sigma,\sigma\nu} - h_{\nu\sigma,\sigma\mu} + h_{\sigma\sigma,\mu\nu} + \delta_{\mu\nu}(h_{\varrho\sigma,\varrho\sigma} - h_{\varrho\varrho,\sigma\sigma}) = 0. \tag{10}$$

Introducing the following coordinate conditions

$$h_{\mu\sigma,\sigma} = \frac{1}{2} h_{\sigma\sigma,\mu} \tag{11}$$

the equations (10) simplify to

$$h_{\mu\nu,\varrho\varrho} + \frac{1}{2}\delta_{\mu\nu} h_{\sigma\sigma,\varrho\varrho} = 0. \tag{12}$$

Taking the trace we get

$$\Box h_{\sigma\sigma} = 0. \tag{13}$$

Introducing (13) back into (12), it is seen that not only the trace but all field quantities satisfy the d'Alembert equation

$$\Box h_{\mu\nu} = 0. \tag{14}$$

[3] Compare the Appendix.

The conditions (11) are nothing else but the linearized conditions for harmonic (de Donder) coordinates

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0. \tag{1''}$$

Let us stress once more that the linearized theory is deprived of sense in the traditional theory but becomes meaningful in the modified theory and is closely analogous to the theory of the electromagnetic field. The metric field is analogue of the electromagnetic potentials and the conditions (11) are analogue of the Lorentz condition. Moreover, the curvature tensor $R_{\mu\nu\varrho\sigma}$ constitutes a close analogue of the electromagnetic field stregth. The linearized curvature tensor

$$R_{\mu\nu\varrho\sigma} = \frac{1}{2}(h_{\mu\sigma,\nu\varrho} - h_{\mu\varrho,\nu\sigma} - h_{\nu\sigma,\mu\varrho} + h_{\nu\varrho,\mu\sigma}) \tag{15}$$

is invariant under the following gauge transformations.

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{2}(f_{\mu,\nu} + f_{\nu,\mu}) \tag{16}$$

involving four arbitrary functions f_μ. Since the metric properties of space-time are left unchanged under the transformations (16) these last are interpretable as coordinate transformations.

3. *Conclusion*

In view of the fact that the linearized theory constitutes an admissible approximation to the modified theory (but does not constitute any approximation to the original theory of Einstein)[4] we infer that only the modified gravitational field possesses features similar to other fields, e.g. to the electromagnetic field: it possesses solutions representing waves periodic in time as well as truly limited wave packets $\left(\text{decreasing faster than } \sim\frac{1}{r}\right)$. Only the modified theory of gravitation admits a localization of energy-momentum, and is quantizable. The quanta of the modified gravitational field are gravitons. On the other hand, if quantization of the traditional gravitational field theory is at all possible, the only quantum state compatible with Einstein's postulate of General Relativity is a graviton-vacuum.

APPENDIX[5]

Non-existence of a linear approximation for Einstein's theory.

In the early stage of the development of Einstein's theory it was believed that a transition to the case of a weak gravitational field (*in vacuo*) is possible. Writing the metric tensor components in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

[4] As shown in the Appendix.

[5] The contents of the Appendix was prepared in collaboration with Miss B. Gorczyca.

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where $h_{\mu\nu}$ and their derivatives are small so that it makes sense to omit higher powers of $h_{\mu\nu}$ it was shown that, by a suitable choice of the system of coordinates, it is possible to reduce $h_{\mu\nu}$ to the transversal-traceless part h_{kl}^{TT}. The Lagrangian simplifies to

$$\mathcal{L} = \frac{1}{2} (h_{il,\varrho}^{TT})^2 \tag{2}$$

wherefrom the wave equation

$$\Box h_{ij}^{TT} = 0 \tag{3}$$

follows. In this approximation the theory may be put into a canonical form with the Hamiltonian density

$$\mathcal{H} = -\tau_0^0 = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}^{TT}} \dot{h}_{ij}^{TT} - \mathcal{L}. \tag{4}$$

Thus, the component τ_0^0 of Einstein's quasi-tensor appears as if it were a well localized quantity ,not representable in the form of a 3-divergence which contradicts the well known facts about Einstein's theory. Therefore something must be wrong with this linear approximation. In order to show why it is not at all an approximation to any exact solution of Einstein's theory let us discuss it in detail.

For this discussion it is convenient to introduce orthochronous coordinates

$$g_{0k} = 0, \quad k = 1, 2, 3. \tag{5}$$

The Lagrangian of the gravitational field simplifies to

$$\mathcal{L} = \frac{1}{4} \{g_{00}^{-1/2} A + g_{00}^{1/2} B + (g_{00}^{1/2})_{,k} C^k\} \tag{6}$$

where

$$A = \frac{1}{4} \sqrt{-g_3}\, \dot{g}_{ij} \dot{g}_{kl} (g^{ik} g^{jl} - g^{ij} g^{kl}), \tag{6'}$$

$$B = \frac{1}{4} \sqrt{-g_3}\, g_{kl,i} g_{rs,j} \{g^{ij} (g^{ks} g^{lr} - g^{kl} g^{rs}) + 2 g^{kj} (g^{il} g^{rs} - g^{ir} g^{ls})\} \tag{6''}$$

$$C^k = \sqrt{-g_3}\, g_{ij,l} (g^{il} g^{jk} - g^{ij} g^{kl}), \tag{6'''}$$

with

$$g_3 = \operatorname{Det} g_{kl}. \quad k, l = 1, 2, 3. \tag{7}$$

We notice the following remarkable fact: in orthochronous coordinates (5) the Lagrangian does not involve the time derivative of g_{00}. By varying the Lagrangian with respect to g_{00} we obtain the tenth equation of Einstein

$$G^{00} = 0 \tag{8}$$

which is a purely algebraic equation for g_{00}. Its solution is

$$g_{00} = \frac{A}{B - C^k_{,k}}. \tag{9}$$

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In orthochronous coordinates the Hamiltonian assumes the form

$$\mathcal{H} = -\tau_0^0 = (g_{00})^{-1/2}(A - g_{00}B) - g_{00}^{1/2}{}_{,k}C^k \tag{10}$$

and may be written as

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \tag{11}$$

where $\mathcal{H}_1$ is of the form of a divergence

$$\mathcal{H}_1 = -(g_{00}^{1/2}C^k)_{,k} \tag{12}$$

while $\mathcal{H}_2$ is

$$\mathcal{H}_2 = (g_{00})^{-1/2}(A - g_{00}(B - C^k_{,k})). \tag{13}$$

Introducing g_{00} from (9) into (13) is it seen that $\mathcal{H}_2$ vanishes identically. Thus, in the exact theory the Hamiltonian reduces to $\mathcal{H}_1$ which is of the form of a 3-divergence. On the other hand, in the weak field approximation one used to put the approximative value $g_{00} = 1$ into (13) so that $\mathcal{H}_2$ appears as a non-vanishing quantity

$$\mathcal{H}_2 = A - B + C^k_{,k} \tag{14}$$

and goes over into the Hamiltonian of the linearized theory. In this way it is seen that the results obtained in the linearized theory are only apparent and vanish if one replaces „approximative" solutions by exact solutions.

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