ONE-PARTICLE OPERATORS FOR THE DIRAC PARTICLE

By H. Janyszek and A. Rakowski

Department of Theoretical Physics Nicholas Copernicus University*

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Various schemes of one-particle operators (expressed in terms of the projection operators into the positive and negative energy states) have been defined and their properties investigated. The prominent role of the scheme of Foldy-Wouthuysen within this set has been confirmed.

It is well known that there is some arbitrariness in the choice of the basic observables, such as the position operator, even for the free particles, if we deduce their quantum-mechanical properties from the more general scheme of the field theory (rather than by elementary correspondence arguments). The main reason for this merely that operators related to the energy, momentum and total angular momentum of the particle are well defined as generators of the infinitesimal displacements and rotations, whereas no such simple relation exists for the position operator. It thus remains undetermined. This is true also for other operators depending on it, such as velocity, orbital angular momentum, and spin of the particle (the last operator may be treated as the difference between the total and the orbital angular momenta, excluding the trivial case of the spin-zero particles), unless some additional criterions allow us to select the scheme of the "true" one-particle observables among infinitely many possibilities. For fermion particles this problem is immediately related to the interpretation of the Dirac equation which has been implied by the transformation of Foldy and Wouthuysen [1] and, in particular, to the physical meaning of their "mean position operator". Its distinguished role among the many other position operators which are formally definable (including the traditional $\boldsymbol{x}$) has been successively explained in the papers of Bose, Gamba, Sudarshan [2], Pac [3], [4] and Mathews, Sankaranarayanan [5], [6], [7]. They compared the two new schemes of describing the Dirac particle — those of Foldy-Wouthuysen and of Cini-Touschek [8] (shortly denoted the "FW" and "CT" schemes) — with the traditional "D scheme". Some generalization of these results has been given in our two earlier papers,[1] by taking into account an infinite group of unitary transformation containing the FW- and CT-transformations as the two limiting cases. Hence, there

* Address: Katedra Fizyki Teoretycznej Uniwersytetu M. Kopernika Toruń, ul. Grudziądzka 5, Polska.

[1] Hanus, Janyszek, Rakowski [9] [10], hereafter denoted as (I). All symbols used here are the same as those defined in (I). A more detailed list of references may also be found there.

is a possibility of defining an infinite set of "generalized U-operators", in analogy to the FW-mean operators (denoted in (I) as "U_0-operators"). This has been used for correboraling the distinction of the FW-scheme in problems of physical interpretation under such generalized assumptions.

The choice of the "true" relativistic position observable is an involved and rather controversial problem, widely discussed from various point of view. Every definition of a position observable must specify its properties under space-time transformations and its commutation properties as well. In the previously quoted papers [5]–[7] Mathews and Sankaranarayanan have proposed a simple and convincing set of conditions which might be imposed on an acceptable position observable $\mathcal{X}$. They have also shown that the U_0-position operator alone fulfills all these conditions simultaneously. This point of view (extended by a comparision made within the infinite set U-position operators) will be adopted in our present considerations, similarly as has been assumed in (I). Let us summarize these "MS-postulates". Apart from the rather delicate condition of the Lorenz-covariance of $\mathcal{X}$, these authors have maintained the postulate of its proper behaviour under space and time inversions. The commutation properties of $\mathcal{X}$ are

$$[\mathcal{X}_j, p_k] = i\delta_{jk}, \tag{1}$$

$$[\mathcal{X}_j, \mathcal{X}_k] = 0, \tag{2}$$

$$\left[\mathcal{X}, \frac{H}{E}\right] = 0. \tag{3}$$

The formula (3) says that $\mathcal{X}$ is a one particle operator, separately defined on positive and negative energy eigenstates. The complete equivalency of (3) and the relation

$$\mathcal{X} = \frac{\boldsymbol{p}}{E}\frac{H}{E}, \tag{4}$$

(previously assumed in [5]–[7] as an independent condition) has been shown in (I).

Depending on the order in which these four conditions are successively introduced, various kinds of "intermediary" operators, only partially satisfying the MS-conditions, may be defined. The properties of these operators (and of the schemes of quasi-observables constructed in terms of the former) may be interesting in many respects. Therefore, as an extension of the results obtained in (I), we now propose to change the order of MS-condition succession. It is different than that established by these authors and that proposed in (I). On the other hand, we start from the same set of generalized U-operators $\Omega_{U'}^{(U)}$ which have been introduced in (I). The freedom of expressing all operators in any picture within the set U is manifested by the subscript U (symbols without any subscript refer to the traditional D-picture). The definition of the U-position operators

$$\boldsymbol{X}_U^{(U)} = \boldsymbol{x} \tag{5}$$

makes it obvious that $\boldsymbol{X}_{U'}^{(U)}$ fulfill all MS-conditions except one, namely (3). Because of this has been selected in (I).

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Now, we begin by postulating (3). The simplest way of constructing one-particle operators for the Dirac particle is by means of projecting the operators into the subspaces belonging to the positive and negative energy states,

$$\Lambda^{\pm}_{U'} = \frac{1}{2}\left(I \pm \frac{H_{U'}}{E}\right). \tag{6}$$

For an arbitrary operator $A_{U'}$, we introduce the symbol

$$A^{\pm} = \Lambda^{\pm}_{U'} A_{U'} \Lambda^{\pm}_{U'}. \tag{7}$$

Hence

$$\tilde{A} = A^{+}_{U'} + A^{-}_{U'}. \tag{8}$$

is the "one-particle part" of $A_{U'}$ ("observable projection" according to Pryce [11]) while

$$A_{U'} = \tilde{A}_{U'} + \Lambda^{+}_{U'} A_{U'} \Lambda^{-}_{U'} + \Lambda^{-}_{U'} A_{U'} \Lambda^{+}_{U'}. \tag{9}$$

$A_{U'}$ is a one-particle operator (*e. g.* $A_{U'} = \tilde{A}_{U'}$) only when

$$\left[A_{U'}, \frac{H_{U'}}{E}\right] = 0. \tag{10}$$

This is easily verified by transforming $\tilde{A}_{U'}$ into the form

$$\tilde{A}_{U'} = \frac{1}{2}\left(A_{U'} + \frac{H_{U'}}{E} A_{U'} \frac{H_{U'}}{E}\right) = A_{U'} + \frac{1}{2}\left[\frac{H_{U'}}{E}, A_{U'}\right]\frac{H_{U'}}{E}. \tag{11}$$

Using (8) and (11) we calculate one-particle parts, first, the traditional operators. Elementary calculations give

$$\tilde{\boldsymbol{p}}_{U'} = \boldsymbol{p}_{U'} = \boldsymbol{p}, \tag{12}$$

$$\tilde{\boldsymbol{\alpha}}_{U'} = \frac{\boldsymbol{p}}{E}\frac{H_{U'}}{E}, \tag{13}$$

$$\tilde{\beta}_{U'} = \frac{m}{E}\frac{H_{U'}}{E}, \tag{14}$$

$$(\tilde{\gamma}_5)_{U'} = -\sigma_p\left[\frac{m}{E}\sin f' + \frac{p}{E}\cos f'\right]\frac{H_{U'}}{E}, \tag{15}$$

$$\tilde{x}'_{U'} = \boldsymbol{x} + \frac{im}{2pE^2}(p\cos f' - m\sin f')\beta\boldsymbol{\alpha} + \frac{1}{2pE^2}\left[mp - pE^2\frac{df'}{dp} - \right.$$
$$\left. - m(p\cos f' - m\sin f')\right]\beta\boldsymbol{\alpha}_{\|} - \frac{1}{2pE}[E^2 - m(m\cos f' + p\sin f')]\boldsymbol{\sigma}\times\frac{\boldsymbol{p}}{p}, \tag{16}$$

$$\tilde{\boldsymbol{l}}_{U'} = \boldsymbol{l} + \frac{im}{2E^2}\left(p\cos f' - m\sin f'\right)\beta\left(\boldsymbol{\alpha}\times\frac{\boldsymbol{p}}{p}\right) + \frac{1}{2E^2} +$$
$$+ [E^2 - m(m\cos f' + p\sin f')\Big)\Big]\boldsymbol{\sigma}_{\perp}, \tag{17}$$

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$$\tilde{\sigma}_{U'} = \boldsymbol{\sigma} - \frac{im}{E^2}(p\cos f' - m\sin f')\beta\left(\boldsymbol{\alpha}\times\frac{\boldsymbol{p}}{p}\right) - \frac{1}{E^2}[E^2 - m(m\cos f' + p\sin f'])\boldsymbol{\sigma}_\perp, \tag{18}$$

$$\tilde{\boldsymbol{j}}_{U'} = \boldsymbol{l}_{U'} + \frac{1}{2}\boldsymbol{\sigma}_{U'} = \boldsymbol{j}, \tag{19}$$

$$(\tilde{\sigma}_p)_{U'} = \sigma_p, \tag{20}$$

$$(\boldsymbol{\alpha}_{U'})_\perp = 0 \tag{21}$$

In a similar way the one-particle parts of the generalized U-operators $\Omega_{U'}^{(U)}$ (whose explicit forms are given in (I)) may be calculated. We have

$$\tilde{\boldsymbol{X}}_{U'}^{(U)} = \boldsymbol{x} + \frac{i}{2pE^2}[(p\sin f + m\cos f)(p\cos f' - m\sin f')]\beta\boldsymbol{\alpha} + $$
$$+ \frac{i}{2pE^2}\left[mp - pE^2\frac{df'}{dp} - (p\sin f + \cos f)(p\cos f' - m\sin f')\right]\beta\boldsymbol{\alpha}_{||} +$$
$$+ \frac{1}{2pE^2}[E^2 - (m\cos f + p\sin f)(m\cos f' + p\sin f')]\boldsymbol{\sigma}\times\frac{\boldsymbol{p}}{p}, \tag{22}$$

$$\tilde{\boldsymbol{L}}_U^{(U)} = \boldsymbol{l} + \frac{i}{2E^2}[(p\sin f + m\cos f)\,(p\cos f' - m\sin f')]\,\beta\left(\boldsymbol{\alpha}\times\frac{\boldsymbol{p}}{p}\right) +$$
$$+ \frac{1}{E^2}[E^2 - (m\cos f + p\sin f)\,(m\cos f' + p\sin f')]\boldsymbol{\sigma}_\perp, \tag{23}$$

$$\tilde{\boldsymbol{\Sigma}}_{U'}^{(U)} = \boldsymbol{\sigma} - \frac{i}{E^2}[(p\sin f + m\cos f)(p\cos f' - m\sin f')]\beta\left(\boldsymbol{\alpha}\times\frac{\boldsymbol{p}}{p}\right) +$$
$$+ \frac{1}{E^2}[E - (m\cos f + p\sin f)\,(m\cos f' + p\sin f')]\boldsymbol{\sigma}_\perp \tag{24}$$

$$\tilde{\boldsymbol{J}}_{U'}^{(U)} = \boldsymbol{J} = \boldsymbol{j}, \tag{25}$$

$$(\tilde{\Sigma}_p)_{U'}^U = \sigma_p, \tag{26}$$

$$\tilde{\boldsymbol{V}}_{U'}^{(U)} = \frac{\boldsymbol{p}}{E}\frac{H_{U'}}{E} = \boldsymbol{\alpha}_{U'}. \tag{27}$$

The results (25), (26), (27) do not depend on the superscript U.

The formulae (12), (19), (20), (25) and (26) corroborate the one-particle character of the operators $\boldsymbol{p}$, $\boldsymbol{j}$ and σ_p. These are the same operators which have proved to be invariant under the group of transformations U' and identical with the respective sets of the generalized U-operators (for details see (I)). The fourth such operator $\boldsymbol{\alpha}_\perp$ possesses, in accord once with (21), the vanishing one-particle part. Several properties of some of the operators (12)–(21), expressed in the picture $U' = I$, have already been discussed briefly by Pryce [11].

As might be expected, the formulae (16), (17) and (18), are special cases of the more general ones (22), (23) and (24), respectively, and may obtained from the latter by putting $U = I$ (*e. g.* $f' = 0$). On the other hand, a rather unexpected result is that expressed by

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(13) and (27): the one-particle part of the Dirac operator $\boldsymbol{\alpha}_{U'}$ becomes identical with the FW mean-velocity operator $\boldsymbol{V}^{0}_{U'}$ (the latter being identical, in turn, with the one-particle parts of all U-generalized velocity operators $\boldsymbol{V}^{(U)}_{U'}$, independent of U). The immediate connection of this result with (21) is obvious.

The most interesting conclusions are those concerning the properties of the set of the position operators $\tilde{\boldsymbol{X}}^{(U)}_{U'}$, given by (22). They have been defined as one-particle operators, so they all automatically fulfil the MS-condition (3)(hence, also (4)). The explicit form of (22) immediately shows that the condition (1) is fulfilled, too, and that all $\tilde{\boldsymbol{X}}^{(U)}_{U'}$ are polar vectors, invariant under time inversion. The only MS-condition not fulfilled by all members of the set (22) is then (2). In consequence, the respective operators $\tilde{\boldsymbol{L}}^{(U)}_{U'}$ and $\tilde{\Sigma}^{(U)}_{U'}$ do not possess, in general, correct commutation properties of angular momentum operators. This deficiency is the main hindrance in applying these operators as one-particle observables. Putting $U = U_{\infty}$ (the CT-scheme) we find, as a special case, the set of operators previously defined and discussed in [2] and [4].

The already known properties of $\Omega^{(U)}_{U'}$ us the corresponding operator becomes identical with the FW-mean-position operator $\boldsymbol{X}^{(0)}_{U'}$ which possesses commuting components, only when $U = U_0$ the FW-scheme. This conclusion may be verified immediately. Putting, for simplicity, $U = U'$ (as calculations may be performed in any picture U') we obtain, after somewhat lengthy, though elementary, calculations

$$[(\tilde{X}^{(U)}_{U})_j, (\tilde{X}^{(U)}_{U})_k] = \frac{1}{2p^2}\left(\frac{d}{dp}\,\frac{A_1}{E}\right)\frac{A_2}{E}\,\beta(\boldsymbol{\alpha}\times\boldsymbol{p})_l + \frac{1}{2ip^2}\left(\frac{A_2}{E}\right)^2\sigma_{\parallel l} + \frac{1}{2ip}\left(\frac{d}{dp}\,\frac{A_z}{E}\right)\frac{A_2}{E}\,\sigma_{\perp l}, \tag{28}$$

where

$$A_1 = m\cos f + p\sin f$$
$$A_3 = p\cos f - m\sin f \tag{29}$$

(j, k, l, denoting a cyclic permutation of 1, 2, 3). The right-hand side of (28) is a linear combination of the three linearly independent vectors $\boldsymbol{\sigma}_{\perp}$, $\boldsymbol{\sigma}_{\parallel}$ and $\beta(\boldsymbol{\alpha}\times\boldsymbol{p})$, so it vanishes only when $A_2 = 0$. This assumption leads to the FW case (for details see (I)).

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