SUPERCONVERGENT SUM RULES FOR MESON-BARYON SCATTERING AND J-PLANE SINGULARITIES

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The possible superconvergent relations which correspond to the decouplet exchange in the t-channel are considered for meson-baryon scattering. The following combination of scattering amplitudes is used in the analysis of sum rules:

$$T_{10} = T_{\pi^- p} - T_{\pi^+ p} + T_{K^+ p} - T_{K^- p} + T_{\overline{K}^0 p} - T_{K^0 p}.$$

It is found that the superconvergence relations hold for nonspin-flip but fail for the spin-flip amplitudes. It is suggested that the failure of the latter may be due to the j-plane branch point singularity which protects the fixed singularity at the wrong signature nonsense point $j = 0$. It is argued by means of the many channel unitarity condition that this branch point singularity is likely to be absent at t-channel partial wave helicity amplitudes, which are regular at $j = 0$. Thus it does not prevent the sum rule for the nonspin-flip amplitude from holding.

I. Introduction

Much interest has recently been devoted to the study of the superconvergence relations [1–3] and finite energy sum rules (*i. e.* superconvergence relations for scattering amplitudes with subtracted asymptotic part) [3–4].

The superconvergence relations, namely the sum rules of the following form:

$$\int_{-\infty}^{+\infty} \mathrm{Im}\, A(\nu, t)\, d\nu = 0 \tag{1.1}$$

are the rigorous consequences of the analytic properties and (assumed) asymptotic behaviour of the scattering amplitudes, the latter being determined by the complex j-plane singularities in the crossed channel. In this respect, the superconvergence relations and finite energy sum rules are useful tools in testing the assumptions concerning the asymptotic behaviour of scattering amplitudes and, in particular, they may be useful in establishing whether some extra j-plane singularities are important, apart from Regge poles associated with the well-established resonances.

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The leading j-plane singularities which, in particular, control the asymptotic behaviour of meson-baryon scattering amplitudes are: the Pomeranchuk pole and the Regge poles associated with nonets of vector and tensor mesons which are assigned to the singlet and octet representations of SU (3). The absence of meson resonances which could be assigned to the 27-plet representation of SU(3) has led to the conjecture that the corresponding j-plane singularity α_{27} (0) has negative intercept

$$\alpha_{27}(0) < 0 \tag{1.2}$$

and, as a consequence of the condition (1.2), we receive the familiar superconvergence relation for the corresponding meson-baryon spin flip amplitude B_{27}:

$$\int_0^\infty \mathrm{Im}\, B_{27}(\nu, t)\, d\nu = 0, \tag{1.3}$$

which has been discussed by several authors [5] and found to be quite well satisfied. It is worthwhile to note, however, as has been pointed out in Ref. [6], that the condition (1.2) seems to be incompatible with the possible appearance of the cut in the complex j-plane generated by the exchange of two vector Regge poles. In this respect the remarkable success of the sum rule (1.3) suggests that this particular singularity is, for some reasons, unimportant.

The sum rule (1.3) is the only nontrivial superconvergence relation for meson-baryon scattering which can be obtained under the rather weak assumption (1.2). If, however, one makes the conjecture that the respective j-plane singularity α_{10} (0) associated with the decouplet exchange in the t-channel satisfies the condition

$$\alpha_{10}(0) < -1, \tag{1.4}$$

which is supported by the lack of convicing experimental evidence concerning meson resonances which could be assigned to the decouplet representation of SU(3), one obtains two more superconvergence relations:

$$\int_0^\infty \mathrm{Im}\, A'_{10}(\nu, 0)\, d\nu = 0 \tag{1.5a}$$

$$\int_0^\infty \mathrm{Im}\, \nu B_{10}(\nu, 0)\, d\nu = 0 \tag{1.5b}$$

where

$$A'_{10} = A_{10} + \nu B_{10}$$

and A_{10} and B_{10} are the conventional meson-baryon invariant scattering amplitudes and the subscript refers to the t-channel SU(3) representation. Let us remark that the condition (1.4) seems to be a not very stringent one, since similar assumptions underly the derivation of most finite energy sum rules.

The sum rule (1.5a) has recently been analysed [7–9] and found to be very well satisfied. In the recent note [10] we questioned, however, whether the sum rules (1.5a, b) hold separately and in this paper we wish to discuss this problem in more detail, with special emphasis on the connection of these sum rules with the possible t-channel j-plane singularities.

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In Section II we discuss the sum rules (1.5a, b) taking the Barger–Rubin [11] combination of meson-bayron scattering amplitudes:

$$T_{10} = T_{\pi^-p} - T_{\pi^+p} + T_{K^+p} - T_{K^-p} + T_{\bar{K}^0p} - T_{K^0p}$$

as the representative for the decouplet exchange amplitude T_{10} and find that in contrast to the sum rule (1.5a), the sum rule (1.5b) fails to be satisfied.

We suggest that the failure of this sum rule may be due to the cut in the complex j-plane which protects the fixed one-over square root singularity at wrong signature nonsense point $j = 0$. We confirm the existence of this fixed singularity by computing its residuum defined by the integral

$$b_0 = \int_0^\infty \mathrm{Im}\, B_{10}(\nu, 0) d\nu.$$

In section III we analyse, by simple generalization of the one channel case, the role played by cuts in complex j-plane in establishing the mutual consistency between the fixed singularities at wrong singnature nonsense points and the nonlinear many channel unitarity condition. The main conclusion of this section is that cuts in the complex j-plane associated with fixed singularities at wrong signature nonsense points may be absent at partial wave helicity amplitudes which are regular at those points. Conversely, they must appear at amplitudes singular at these points.

In this way the branch point singularity associated with the wrong signature nonsense point $j = 0$ may not prevent the sum rule (1.5a) from holding but leads to the failure of its companion (1.5b).

Finally, in Section IV we give the summary of results.

II. The decouplet exchange superconvergence relations

In this section we discuss the possible superconvergence relations for this part of the meson-baryon scattering amplitude which corresponds to the decouplet exchange in the t-channel.

Let us introduce the respective amplitude T_{10}:

$$T_{10} = -A_{10}(s, u; t) + i\gamma \cdot \left(\frac{q_1+q_2}{2}\right) B_{10}(s, u; t)$$

$$A'_{10} = A_{10} + \nu B_{10}$$

$$\nu = \frac{s-u}{4M} \tag{2.1}$$

where s and u are the usual invariant variables, M is the baryon mass and $q_{1,2}$ denote the four momenta of mesons in the initial and final state respectively; the subscript refers to the t-channel SU(3) representation. In what follows we shall limit ourselves to $t = 0$.

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The invariant amplitudes A'_{10} and B_{10} satisfy the following dispersion relations:

$$A'_{10}(\nu; 0) = \frac{1}{\pi} \int_0^\infty \operatorname{Im} A'_{10}(\nu', 0) \left[\frac{1}{\nu' - \nu} - \frac{1}{\nu' + \nu}\right] d\nu'$$

$$B_{10}(\nu; 0) = \frac{1}{\pi} \int_0^\infty \operatorname{Im} B_{10}(\nu'; 0) \left[\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu}\right] d\nu' \tag{2.2}$$

where the respective pole terms are also included in the dispersive integrals.

The asymptotic behaviour of invariant amplitudes A'_{10} and B_{10} is determined by the j-plane singularities in the t-channel and has the following form:

$$A'_{10}(\nu; 0) \sim \nu^{\alpha_{10}(0)} \tag{2.3}$$

$$B_{10}(\nu; 0) \sim \nu^{\alpha_{10}(0)-1} \tag{2.4}$$

where $\alpha_{10}(0)$ denotes the intercept of the corresponding j-plane singularity. If $\alpha_{10}(0)$ satisfies the condition:

$$\alpha_{10}(0) < -1$$

which is supported by the lack of experimental evidence concerning meson Regge trajectory with the quantum numbers characteristic for the decouplet, we obtain the following independent superconvergence relations:

$$a_0 = \int_0^\infty \operatorname{Im} A'_{10}(\nu; 0)\, d\nu = 0 \tag{2.5}$$

$$b_1 = \int_0^\infty \operatorname{Im} \nu B_{10}(\nu; 0)\, d\nu = 0. \tag{2.6}$$

It is worthwhile to confront these relations with the conditions for the absence of fixed singularities of t-channel partial wave helicity amplitudes at right signature nonsense point $j = -1$. The corresponding Gribov–Froissart representation for t-channel partial wave helicity amplitudes has the following form [12]:

$$f^{(-)}_{+;10}(j; t) = \frac{1}{8\pi^2 (pq)^{j+1}} \left\{ -p^2 \int_0^\infty \operatorname{Im} A_{10}(s'; t) \times \right.$$

$$\times Q_j \left(\frac{s' + p^2 + q^2}{2pq}\right) ds' + \frac{M(pq)}{2j+1} \int_0^\infty \operatorname{Im} B_{10}(s', t) \times$$

$$\left. \times \left[(j+1) Q_{j+1} \left(\frac{s' + p^2 + q^2}{2pq}\right) + j Q_{j-1} \left(\frac{s' + p^2 + q^2}{2pq}\right)\right] ds' \right\}$$

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$$f^{(-)}_{-;10}(j;t) = \frac{1}{16\pi^2}\frac{1}{(pq)^j}\frac{[j(j+1)]^{1/2}}{j+1/2} \times \int_0^\infty \mathrm{Im}\, B_{10}(s';t)$$

$$\left[Q_{j-1}\left(\frac{s'+p^2+q^2}{2pq}\right) - Q_{j+1}\left(\frac{s'+p^2+q^2}{2pq}\right)\right] ds' \tag{2.7}$$

where

$$4p^2 = t - 4M^2$$

$$4q^2 = t - 4\mu^2$$

where, due to the crossing properties of A_{10} and B_{10} amplitudes, only odd signature partial waves are different from zero. Comparing (2.5, 2.6) with the representation (2.7) we find that the superconvergence relations (2.5, 2.6) are conditions for vanishing of residua of fixed poles at $j = -1$ which come from the function Q_l. On the other hand, we find from the representation (2.7) that the integral:

$$b_0 = \int_0^\infty \mathrm{Im}\, B_{10}(\nu;0)\, d\nu \tag{2.8}$$

represents the residuum of the fixed singularity of the $f^{(-)}_{-;10}(j;t)$ amplitude at the wrong signature nonsense point $j = 0$, which singularity is expected to be present due to the effect of third double spectral function. (The amplitude $f^{(-)}_{+;10}(j;t)$ is regular at this point).

We analyse the sum rules (2.5, 2.6) taking the following combination[1] of elastic scattering amplitudes as representing the amplitude T_{10}:

$$T_{10} = T_{\pi^- p} - T_{\pi^+ p} + T_{K^+ p} - T_{K^- p} + T_{\bar{K}^0 p} - T_{K^0 p}. \tag{2.9}$$

In this representation the sum rule (2.5) may then be converted into the following form:

$$a_0 = -\frac{\mu_\pi^2}{2M_p^2}\frac{g_{N\pi}^2}{4\pi} + \frac{g_{\Sigma Kp}^2}{4\pi}\frac{[(M_\Sigma - M_p)^2 - \mu_K^2]}{4M_p^2} -$$

$$-\frac{g_{\Lambda Kp}^2}{4\pi}\frac{[(M_\Lambda - M_p)^2 - \mu_K^2]}{4M_p^2} + \int_{\nu_{\Lambda\pi}}^{\mu_K} \mathrm{Im}\,(A'_{\bar{K}^0 p} - A'_{K^- p})\,d\nu +$$

$$+\frac{1}{4\pi^2}\int_{\mu_\pi}^{\infty} (\nu^2 - \mu_\pi^2)^{1/2}[\sigma_{\pi^- p} - \sigma_{\pi^+ p}]\,d\nu +$$

$$+\frac{1}{4\pi^2}\int_{\mu_K}^{\infty} (\nu^2 - \mu_K^2)^{1/2}[\sigma_{K^+ p} - \sigma_{K^- p} + \sigma_{\bar{K}^0 p} - \sigma_{K^0 p}]\,d\nu = 0 \tag{2.10}$$

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[1] This identification requires the couplings of vector Regge poles to be SU(3) symmetric. It is, fortunately, consistent with experiment [13].

where the first three terms come from the nucleon, Σ and Λ poles respectively and $\sigma_{\pi^\pm p}$, $\sigma_{K^\pm p}$, $\sigma_{K^0 p}$, $\sigma_{\bar{K}^0 p}$ are the corresponding total cross-sections. This sum rule has recently been analysed [7–9] and found to be very well satisfied; its success is also supported by the success of individual finite energy sum rules for πN [14] and KN [15] scattering.

We shall now consider the sum rule (2.6). In the representation (2.9) it takes the following form:

$$b_1 = -\frac{\mu_\pi^2}{2M_p^2}\frac{g_{N\pi}^2}{4\pi} + \frac{g_{\Sigma Kp}^2}{4\pi}\left(\frac{M_\Sigma^2 - M_p^2 - \mu_K^2}{4M_p^2}\right) - \frac{g_{\Lambda Kp}^2}{4\pi}\left(\frac{M_\Lambda^2 - M_p^2 - \mu_K^2}{4M_p^2}\right) +$$
$$+ \int_{\nu_{\Lambda\pi}}^{\mu_K} d\nu \, \mathrm{Im}\, \nu[B_{\bar{K}^0 p} - B_{K^- p}] + \int_{\mu_\pi}^{\infty} d\nu \, \mathrm{Im}\, \nu[B_{\pi^- p} - B_{\pi^+ p}] +$$
$$+ \int_{\mu_K}^{\infty} d\nu \, \mathrm{Im}\, \nu[B_{K^+ p} - B_{K^- p} + B_{\bar{K}^0 p} - B_{K^0 p}] = 0. \tag{2.11}$$

In estimation of the $g_{\Sigma Kp}$ and $g_{\Lambda Kp}$ coupling constants we have used their SU(3) values, namely:

$$\frac{g_{\Lambda Kp}^2}{4\pi} = \frac{1}{4\pi}\frac{g_{N\pi}^2}{3}(1+2\alpha)^2$$
$$\frac{g_{\Sigma Kp}^2}{4\pi} = \frac{g_{N\pi}^2}{4\pi}(1-2\alpha)^2 \tag{2.12}$$

and we have put $\alpha = 0.4$. The integral over the unphysical region has been approximated by the contributions of $Y_0^*(1405)$ and $Y_1^*(1385)$ states respectively. Those states contribute in the following way to the integral b_1:

$$R(Y_0^*) = -\frac{1}{4M_p^2}\frac{g_{Y_0^*}^2}{4\pi}[M_{Y_1^*}^2 - M_p^2 - \mu_K^2]$$
$$R(Y_1^*) = \frac{1}{4M_p^2}\frac{g_{Y_1^*}^2}{4\pi}[M_{Y_1^*}^2 - M_p^2 - \mu_K^2] \times$$
$$\times [\{(M_p + M_{Y_1^*})^2 - \mu_K^2\}\{(M_p - M_{Y_1^*})^2 - \mu_K^2 - 2M_p M_{Y_1^*}\}]\frac{1}{6M_{Y_1^*}^2}$$

and we have put the following values for $\frac{g_{Y_0^*}^2}{4\pi}$ and $\frac{g_{Y_1^*}^2}{4\pi}$

$$\frac{g_{Y_0^*}^2}{4\pi} = 0.32$$
$$\frac{g_{Y_1^*}^2}{4\pi} = \frac{1.9}{M_p^2},$$

according to the estimates which have been done in Ref. [16]. The integral over continuum has been approximated by the resonance contributions in the narrow width approximation.

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In this approximation the various resonance states contribute in the following way to the integral b_1:

$$R(l^+;I) = C_I \frac{\Gamma(l^+;I)}{2q^3(l^+;I)} \cdot \frac{W(l^+;I)}{M_p} \nu(l^+;I) \times (l+1)[E(l^+;I)-(l+1)M_p]$$

$$R(I;l^-) = C_I \frac{\Gamma(l^-;I)}{2q^3(l^-;I)} \frac{W(l^-;I)}{M_p} \nu(l^-;I) \times l[E(l^-;I)+lM]$$

$$\nu(l^\pm;I) = \frac{W^2(l^\pm;I)-M_p^2-\mu^2}{2M_p} \tag{2.13}$$

where

$$\Gamma(l^\pm;I),\ W(l^\pm;I),\ q(l^\pm;I),\ E(l^\pm;I)$$

denote the elastic width, mass, centre of mass momentum, and baryon energy corresponding to the given resonance with isospin I, spin $j = l^\pm 1/2$, and parity $P = (-)^{l+1}$ and C_I equals 1/2, $-1/2$, 2/3, and $-2/3$ for the resonances with isospin equal 1 , 0, 1/2, and 3/2 respectively. The contribution of various particle and resonances are collected in Table I, where contributions within SU(3) multiplets, with the same spin and parity have been added together and represented as a single contribution[2]. The resonance parameters have been taken from Ref. [17].

TABLE I

Evaluation of integrals a_0 (formula 2.10) and b_1 (formula 2.11) in the narrow-width resonance approximation

Patricle or resonance states included	J^P	SU(3) multiplicity	Contribution to:	
			a_0	b_1
$N_\alpha(950)$, $\Lambda_\alpha(1115)$ $\Sigma_\alpha(1190)$	$1/2^{(+)}$	8	+0.8	—0.6
$N_\beta(1570)$, $\Lambda_\beta(1405)$ $\Lambda_\beta(1670)$	$1/2^{(+)}$	$8 \oplus 1$	—0.4	negligible
$\Delta_\delta(1236)$, $\Sigma_\delta(1385)$	$3/2^{(+)}$	10	—0.7	+2.0
$N_\gamma(1530)$, $\Sigma_\gamma(1660)$, $\Lambda_\gamma(1690)$, $\Lambda_\gamma(1520)$	$3/2^{(-)}$	$8 \oplus 1$	+0.2	+0.7
$N_\alpha(1688)$, $\Sigma_\alpha(1910)$, $\Lambda_\alpha(1820)$	$5/2^{(+)}$	8	+0.1	+0.7
$N_\beta(1670)$, $\Sigma_\beta(1767)$, $\Lambda_\beta(1827)$	$5/2^{(-)}$	8	+0.5	—1.9
$\Delta_\delta(1920)$, $\Sigma_\delta(2035)$	$7/2^{(+)}$	10	—0.6	+1.9
$N_\gamma(2190)$, $\Lambda_\gamma(2100)$	$7/2^{(-)}$	8	+0.2	+0.6

[2] In Table I only those states are included for which experimental information is sufficient to build up the respective SU(3) multiplets [18]. The main conclusion namely the failure of the sum rule for the spin-flip amplitude is not altered if we include the remaining resonant state contributions and it turns out that the failure is even more significant.

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We see from this table that the sum rule (2.6) fails to be satisfied at least if saturated by the currently known resonances [17]. The leading baryonic recurrences α, γ octets and δ decouplets all contribute with the same positive sign to the integral b_1, the only negative contribution of the $1/2^+$ baryon octet and $5/2^-$ β nonet being very inadequate to cancel the former. This fact is not accidental but is a reflection of the situation which exists if we analyse the sum rule within the SU(3) symmetry. Using the SU(3) crossing matrices [19], we obtain the following expression for T_{10} in terms of direct channel SU(3) invariant amplitudes:

$$T_{10} = -\frac{9}{40} T^{27} + \frac{1}{4}(T^{10}+T^{\overline{10}}) - \frac{2}{5} T^{8_{11}} + \frac{1}{8} T^{1}. \tag{2.14}$$

The respective baryon multiplets contribute in the following form to the integral b_1:

$$R(l^{\pm}; n) = \frac{\Gamma(l^{\pm}; n)}{2q^3(l^{\pm}; n)} \frac{W(l^{\pm}; n)}{M} \times \nu(l; n)(j+1/2)[E(l^{\pm}; n) \pm (j+1/2)M] \tag{2.15}$$

and we find, comparing (2.15) and (2.14) that the α and γ octets and δ decouplets add constructively in the integral b_1. On the other hand, if we analyse the sum rule (2.5) (still within the SU(3) symmetry) we find from (2.14) that different baryonic multiplets contribute with the mutually changing sign (Im $A'^{(n)}$ is positive by virtue of the optical theorem), offering the possibility that the sum rule (2.5) holds. For the sake of comparison, we have calculated the integral a_0 in the narrow width approximation and the results of calculations are presented in Table I. We see from this table that the contributions coming from different baryonic multiplets tend to cancel each other and the careful analysis of the total cross-section sum rule (2.10), which has been done in Refs [8, 9], has shown that this sum rule is indeed satisfied.

We have also estimated, using the narrow width approximation the integral b_0

$$b_0 = \int_0^{\infty} \operatorname{Im} B_{10}(\nu, 0) d\nu$$

which, as we have remarked above, represents the residuum of fixed singularity of $f^{(-)}_{-;10}$ (j, t) amplitude at wrong signature nonsense point $j = 0$. The possible sum rule:

$$b_0 = \int_0^{\infty} \operatorname{Im} B_{10}(\nu; 0) d\nu = 0$$

should hold [20], provided the effect of the third double spectral functions could be neglected. The various contributions to the integral b_0 are collected in Table II and we see that there is no evidence of the possible sum rule holding and that the partial wave amplitude has the fixed one-over-square root singularity at $j = 0$.

The occurrence of such a singularity at strong interaction amplitudes may be compatible with the strong nonlinear unitarity condition, provided this singularity is shielded in a well defined way by the moving cut in the complex j-plane [21–22]. The corresponding branch point $j_c(t)$ passes through $j = 0$ at respective two particle threshold and may remain at $-1 \lesssim j_c(0) < 0$ for $t = 0$.

TABLE II

Evaluation of the integral $b_0 = \int_0^\infty \mathrm{Im}\, B_{10}\,(\nu, 0)d\nu$ in the narrow width resonance approximation

Particle or resonance states included	J^P	SU(3) multiplicity	Contribution to b_0 in GeV^{-1}
$N_\alpha(950)$, $\Lambda_\alpha(1115)$, $\Sigma_\alpha(1190)$	$1/2^{(+)}$	8	+7.4
$N_\beta(1570)$, $\Lambda_\beta(1405)$, $\Lambda_\beta(1670)$	$1/2^{(-)}$	$8 \oplus 1$	−0.1
$\Delta_\delta(1236)$, $\Sigma_\delta(1385)$	$3/2^{(+)}$	10	+6.6
$N_\gamma(1530)$, $\Sigma_\gamma(1660)$, $\Lambda_\gamma(1690)$, $\Lambda_\gamma(1590)$	$3/2^{(-)}$	$8 \oplus 1$	+0.7
$N_\alpha(1688)$, $\Sigma_\alpha(1910)$, $\Lambda_\alpha(1820)$	$5/2^{(+)}$	8	+0.9
$N_\beta(1670)$, $\Sigma_\beta(1767)$, $\Lambda_\beta(1827)$	$5/2^{(-)}$	8	−1.8
$\Delta_\delta(1920)$, $\Sigma_\delta(2035)$	$7/2^{(+)}$	10	+1.2
$N_\gamma(2190)$, $\Lambda_\gamma(2100)$	$7/2^{(-)}$	8	+0.1

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It appears, therefore, an interesting possibility that the failure of the sum rule (2.6) may be due to this branch point singularity and that the sum rule has to be converted into the form of the finite energy sum rule:

$$\int_0^{\bar{\nu}} \mathrm{Im}\nu B_{10}(\nu; 0)d\nu = \int^{j_c(0)} dj\bar{\nu}^{j+1}\Delta f^c_{-;10}(j; 0)$$

where $j_c(0)$ and $\Delta f^c_{-;10}$ denote the position of the branch point singularity and the discontinuity of the amplitude $f_{-;10}(j, 0)$ when passing across the cut in the j-plane. In the next section we argue that this singularity is likely to be absent at amplitude $f_{+;10}(j, t)$ and thus does not prevent the sum rule (2.5) from holding since the asymptotic behaviour of the amplitude $A'_{10}(\nu; 0)$ is controlled by the singularities of $f_{+;10}(j, t)$ alone (see [12]).

III. Many channel unitarity condition and Regge cut discontinuities

It has recently been realised [22] that it is very useful to analyse the role of cuts in the complex j-plane in removing the inconsistency between the existence of fixed singularities of partial wave amplitudes at wrong signature nonsense points and the two particle (nonlinear) unitarity condition quite apart from the fact that these cuts are generated by many particle intermediate processes [23]. It has been established [21, 22] that, instead of Gribov-Pomeranchuk's essential singularity [24], the partial wave amplitudes have fixed poles at wrong signature nonsense points (which do not contribute to the asymptotic behaviour of scattering amplitudes), which are protected by the moving cuts in the j-plane in a well-defined way so that they do not conflict with the unitarity condition. In this section we wish to point out that some new features may appear if we generalize these considerations to the many channel case.

Let us now first recall how the complex j-plane cuts interfere with the elastic unitarity condition in the one channel case of elastic scattering of spinless particles [22].

The unitarity condition wich determines the right hand cut discontinuity of the partial wave amplitude has the following form:

$$f^{(\pm)}(j;t^+)-f^{(\pm)}(j;t^-) = 2i\varrho(j;t)f^{(\pm)}(j;t^+)f^{(\pm)}(j;t^-). \tag{3.1}$$

It is clear that, owing to its nonlinear structure, this condition rules out the fixed pole as the allowed singularity of the partial wave amplitudes. On the other hand, the left hand cut discontinuity of the amplitude $f^{(\pm)}(j;t)$ have such poles (*e.g.*[25]) located at wrong signature nonsense points (negative odd/even integers for even/odd signature amplitudes, respectively) due to the occurrence of the third (ϱ_{su}) spectral function. If this fact is confronted with the unitarity condition (3.1), one results in the Gribov–Pomeranchuk [24] essential singularities at those nonsense points.

The situation changes, however, if the moving branch points in the complex j-plane are introduced. These branch points $j_c(t)$ induce the j-dependent branch points $t_c(j)$ in the t-plane which, for j sufficiently small (*i.e.* near the wrong signature nonsense points), enter the physical sheet. In this case the unitarity condition no longer determines the right hand cut discontinuity, which is now expressed in the following form (see [22]):

$$f(j;t^+)-f(j;t^-) = 2i\varrho(j;t)[f(j;t^+)-2i\Delta f^c(j;t)]f(j;t^-)+2i\Delta f^c(j;t) \tag{3.2}$$

where $\Delta f^c(j,t)$ denotes the discontinuity of the partial wave amplitude when passing across the cut associated with the branch point $t_c(j)$. If the branch point $t_c(j)$ is introduced, then the fixed pole of the left hand cut discontinuity is no longer a problem; it remains as the fixed pole of the partial amplitude as the whole provided the $t_c(j)$ and $\Delta f^c(j;t)$ satisfy the following conditions

$$t_c(j_n) = 4\mu^2$$

$$\lim_{j\to j_n}(j-j_n)\,[f(j;t^+)-2i\Delta f^c(j;t)] = 0,$$

i.e. the cut associated with the branch point $t_c(j)$ covers completely the unitarity cut if j passes through the wrong signature integer and the discontinuity $\Delta f^c(j;t)$ is singular at this point and cancels the singular part of partial wave amplitude.

We shall now generalise these considerations into the many chanel scattering of particles with spin. Let us introduce the respective partial wave helicity amplitude $\overset{(s)}{f}_{\lambda_1^\alpha\lambda_2^\alpha;\lambda_1^\beta\lambda_2^\beta}(j,t)$ with definite signature "s" where α, β label the respective two-particle channels and $\lambda^\alpha_{1,2}$, $\lambda^\beta_{1,2}$, the corresponding helicities. It is well known that [26] the left hand cut discontinuities of partial wave helicity amplitudes $\overset{(s)}{f}_{\lambda_1^\alpha\lambda_2^\alpha;\lambda_1^\beta\lambda_2^\beta}(j;t)$ have fixed singularities (one-over-square root and fixed pole-type for nonsense-sense or sense-nonsense and nonsense-nonsense

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transitions respetively; the wrong signature nonsense points for the given partial wave helicity amplitude $f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j;t)$ are located at $j_n = \lambda-1-n$, j_n odd/even integer for even/odd signature respectively and λ is defined as below:

$$\lambda = \max\{|\lambda^\alpha_1-\lambda^\alpha_2|,\ |\lambda^\beta_1-\lambda^\beta_2|\}.$$

If these fixed singularities are confronted with the many channel (nonlinear) unitarity condition:

$$f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j,t^+) - f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j,t^-) = 2i\sum_{\gamma;\lambda^\gamma_1\lambda^\gamma_2}\theta(t-t_\gamma) f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\gamma_1\lambda^\gamma_2}(j,t^+)\times$$
$$\times\varrho(j,\gamma;t) f^{(s)}_{\lambda^\gamma_1\lambda^\gamma_2;\lambda^\beta_1\lambda^\beta_2}(j,t^-) \tag{3.3}$$

one still obtains the essential singularities of partial wave helicity amplitudes at these points, which may now be located in the right half j-plane [27]. This difficulty may be avoided if we introduce cuts in the complex j-plane which induce additional branch points $t_c(j;j_n)$ in the t-plane and we have stressed the fact that these branch points, despite their j-dependence, are associated with definite wrong signature nonsense points j_n. In the presence of these additional branch points, the unitarity condition (3.3) has to be modified and, in analogy with (3.2), we obtain the following expression for the right hand cut discontinuities of partial wave helicity amplitudes:

$$f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j,t^+) - f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j,t^-) = 2i\sum_{\gamma;\lambda^\gamma_1\lambda^\gamma_2}\theta(t-t_\gamma)[f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\gamma_1\lambda^\gamma_2}(j,t^+) -$$
$$-2i\Delta f^{c}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\gamma_1\lambda^\gamma_2}(j,t;j^s_n)]\,\varrho(j,\gamma;t)\times f^{(s)}_{\lambda^\gamma_1\lambda^\gamma_2;\lambda^\beta_1\lambda^\beta_2}(j,t^-) + 2i\Delta f^{c}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j,t;j^s_n), \tag{3.4}$$

where the functions $\Delta f^{c}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j;t;j^s_n)$ denote the discontinuities of the respective amplitudes $f^{(s)}_{\lambda^\alpha_1\lambda^\alpha_2;\lambda^\beta_1\lambda^\beta_2}(j;t)$ in passing across cuts associated with the branch points $t_c(j;j^s_n)$. If these branch points are present, then the fixed singularities of the left hand cut discontinuities may remain as the fixed singularities of the partial wave amplitudes (they do not contribute to the asymtotic behaviour (*e.g.* [26]) no matter how far to the right they are located), provided the following conditions are fulfilled by the branch points $t_c(j;j^s_n)$:

i) the cuts associated with the branch points $t_c(j;j^s_n)$ have to cover exactly the respective two-particle threshold cuts.

ii) the discontinuities $\Delta f^{c}_{\lambda^\alpha_1\ldots\lambda^\beta_2}(j,t;j_n)$ must be singular at $j=j_n$ (for amplitudes $f_{\lambda^\alpha_1\ldots\lambda^\beta_2}(j;t)$ singular at $j=j_n$) and must cancel the singular part of the respective amplitudes. If these conditions are fulfilled, then both sides of the Eq. (3.4) behave in the same way for $j\cong j_n$ *i.e.* they are regular and behave like $(j-j)^{-1/2}$ and $(j-j_n)^{-1}$ for *ss*, *ns*, and *nn* transitions respectively. This may be seen most easily if we rewrite the condition (3.3) in a more compact way, keeping only those terms on the right hand side which lead into trouble if the fixed (irregular) singularities (one-over-square root and poles for *sn*/*ns* and *nn*

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amplitudes) are confronted with this condition. Keeping only those terms we obtain:

$$f_{ss}(j, t^+)-f_{ss}(j, t^-) \cong 2i \sum_n f_{sn}(j, t^+)\, \varrho(j, t) f_{sn}(j, t^-)$$

$$f_{sn}(j, t^+)-f_{sn}(j, t^-) \cong 2i \sum_{\bar{n}} f_{s\bar{n}}(j, t^+)\, \varrho(j, t) f_{\bar{n}n}(j, t^-)$$

$$f_{nn}(j, t^+)-f_{nn}(j, t^-) \cong 2i \sum_{\bar{n}} f_{n\bar{n}}(j, t^+)\, \varrho(j, t) f_{\bar{n}n}(j, t^-),$$

where all irrelevant indices have been omitted. We see from these equations that if we allow the fixed singularities to be present then on the right hand side we obtain extra (or higher order) fixed j-plane singularities in comparison with the left hand side, and this is the main source of trouble. On the other hand, if the cuts are introduced, then the Eq. (3.4) has the form:

$$f_{ss}(j, t^+)-f_{ss}(j, t^-) \cong 2i \sum_n [f_{sn}(j, t^+)-$$
$$-2i\Delta f^c_{sn}(j, t; j_n)]\, \varrho(j, t) f_{ns}(j, t^-)+2i\Delta f^c_{ss}(i, t; j_n)$$
$$f_{sn}(j, t^+)-f_{sn}(j, t^-) \cong 2i \sum_{\bar{n}} [f_{s\bar{n}}(j, t^+)-$$
$$-2i\Delta f^c_{s\bar{n}}(j, t; j_n)]\, \varrho(j, t) f_{\bar{n}n}(j, t^-)+2i\Delta f^c_{sn}(i, t; j_n)$$
$$f_{nn}(j, t^+)-f_{nn}(j, t^-) \cong 2i \sum_{\bar{n}} [f_{n\bar{n}}(j, t^+)-$$
$$-2i\Delta f^c_{\bar{n}n}(j, t; j_n)]\, \varrho(j, t) f_{\bar{n}n}(j, t^-)+2i\Delta f^c_{nn}(j, t; j_n) \tag{3.5}$$

and now the fixed singularities of the above structure do not present any trouble *i.e.* both sides of Eq. (3.4) behave in the same way for $j \sim j_n$ provided conditions (*i*) and (*ii*) are fulfilled.

Now we should like to point out the following: the branch point $j_c(t; j_n)$ associated with the definite wrong signature nonsense point j_n must be present at those amplitudes which are singular at $j = j_n$ (*i.e.* at those amplitudes which correspond to sense-nonsense (or nonsense-sense) and nonsense-nonsense transitions for $j = j_n$), in order that the conditions (*i*) and (*ii*) might be satisfied. Conversely, these branch points are not required to occur at those partial wave helicity amplitudes which are regular at $j = j_n$ (*i.e.* at those amplitudes which correspond to sense-sense transitions at $j = j_n$) and the structure of equation (3.5) allows these branch points to be decoupled from regular amplitudes *i.e.* allows the following condition to hold:

$$\Delta f^c_{\lambda_1^\alpha\lambda_2^\alpha;\, \lambda_1^\beta\lambda_2^\beta}(j, t; j_n) = 0 \tag{3.6}$$

for amplitudes regular at $j = j_n$.

It therefore appears very plausible that to the extent that the main role of the moving branch points in the complex j-plane is to remove the inconsistency between the fixed singularities at wrong signature nonsense points and nonlinear unitarity condition, these

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branch points may be absent at those partial wave helicity amplitudes which are regular at these points. Conversely, they must be present at amplitudes which are singular at these points.

IV. Summary of results

We have discussed in Section II the possible superconvergent relations for that part of the meson-baryon scattering amplitude which corresponds to the exchange of the decouplet in the t-channel. We have found that, in contrast to the superconvergence sum rule (2.5) for the non-spin-flip amplitude, the sum rule (2.6) fails to be satisfied. We have suggested that the failure of the latter may be due to the branch point $j_c(t)$ at complex j-plane which protects the fixed singularity of $f^{(-)}_{-;10}(j; t)$ amplitude at $j = 0$. We have argued that this singularity is likely to be absent at the partial wave helicity amplitude which is regular at this point (the amplitude $f^{(-)}_{+;10}(j, t)$ in the case of meson baryon scattering), and it does not prevent the corresponding sum rule (2.5) from holding, since the latter depends on the positions of j-plane singularities of $f^{(-)}_{+;10}(j; t)$ alone. The similar situation which happens for $f_{-;10}$ near $j = 0$ occurs for $f_{+;10}$ near $j = -2$, but the eventual branch point which protects this fixed singularity is located sufficiently far to the left ($j^{(+)}_c(0) \lesssim -2$) and cannot spoil the sum rule (2.5).

Accepting the condition (3.6) to be true, we can easily understand also the success of the superconvergent sum rule corresponding to the 27-plet exchange in the t-channel. By virtue of this condition, the only singularity which should appear in the respective t-channel partial wave helicity amplitudes is the branch point which protects the fixed pole at wrong signature nonsense point, which in this case is located at $j = -1$ and the branch point $j_c(t)$ is expected to appear at $j_c(0) \lesssim -1$ and does not violate the assumption $\alpha_{27}(0) < 0$ which underlies the derivation of the respective sum rule [5]:

$$\int_0^\infty \mathrm{Im}\, B_{27}(\nu, 0)\, d\nu = 0.$$

As a final comment, we should like to point out that cuts in the angular momentum plane, which protect the wrong signature nonsense points singularities, are required to occur not only at strong but also at weak (*i.e.* photoproduction, $\gamma\gamma \to N\bar{N}$ *etc.*) amplitudes. This is due to the fact that, though the unitarity condition is linear for these amplitudes, nevertheless, at the same time, both strong and weak amplitudes are singular at wrong signature nonsense points. This has to be contrasted with the fixed poles of weak amplitudes which occur at right signature nonsense points [28] which do not require any cuts to be protected by since the respective strong interaction amplitudes are regular at those points.

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