# DEGENERATION AND STABILITY OF MAGNETOHYDRODYNAMIC MODES IN A COAXIAL CHANNEL WITH AN HELICAL FIELD

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The eigenfrequency spectrum and stability of perfectly conducting inviscid and incom, pressible fluids confined to cylindrical channels and permeated by twisted magnetic fields is investigated. It is found that even when we limit our investigations to axially symmetric instabilities, solutions can be found that are not stable. This happens for some azimuthal fields $B_{\Phi}(r)$ and for waves longer than a given critical length. In one particular case the flow fields are given, as well as the exact values of the critical wave lengths. In this case the instability is nonconvective and should therefore be easy to detect experimentally.



## 1. Introduction

Many papers treat the problem of stability of ideal conducting fluids in twisted magnetic fields. Most of them start from the assumption that the surface can be deformed continuously so as to conserve the total pressure (magnetic plus mechanical). This is the case with the work of Chandrasekhar (1961), and those of Roberts (1956), Murty (1961), and Charaborty (1961), to name a few. However, even when the fluid is contained in a cylinder with fixed walls, the fluid can be unstable. The instability can be nonconvective.

In this paper evaluations of the azimuthal field (or, in other words, of the longitudinal current) and of the wave lengths required to engender instability are given.

The author investigated the case of an uniform field plus a line current flowing down the centre, giving rise to an azimuthal field of intensity $2I/r$, where $I$ is the current intensity and $r$ the distance from the axis of the cylinder (1967). It was found that an infinite spectrum of eigenvalues for the frequency $\omega$ was possible. All these eigenfrequencies tended to the Alvén frequency for $I \rightarrow 0$. The flow was stable with respect to the axially symmetric instability.

In this paper we will start with a general azimuthal field $B_{\Phi}(r)$, and then restrict our considerations to fields of the type $\beta r^n$, where $n$ is any real number. There will be no axially symmetric instabilities for $n \leqslant -1$.

For $n = 1$ the flow field will be found and the instability classified.

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## 2. *The basic equations*

We will consider an ideal conducting fluid of constant density $\varrho$ between two co-axial cylinders of radii $R_1$ and $R_2$. In some cases there will be just one cylinder of radius $R_2$. The fluid is permeated by a constant magnetic field in the direction of the axis, which we will call the $z$ direction. An $r$ dependent current also flows in the $z$ direction, inducing an azimuthal field given by

$$4\pi J_z(r) = 1/r\, d/dr(rB_\Phi) \tag{1}$$

and the whole current is

$$I = \frac{1}{2}(R_2 B_\Phi(R_2) - R_1 B_\Phi(R_1)). \tag{2}$$

In cylindrical coordinates the time independent field is

$$B = (0, B_\Phi(r), Bz). \tag{3}$$

We will now try to investigate the stability of the fluid for given functions $J_z(r)$. We will only consider disturbances of the form $f(r)e^{i(kz+\omega t)}$, corresponding to the *sausage* type in plasma physics.

The linearized equations of m.h.d. together with the immobile wall condition give

$$\varrho \partial \boldsymbol{v}/\partial t = -\nabla p + \boldsymbol{J} \times \boldsymbol{b} + \boldsymbol{j} \times \boldsymbol{B} \tag{4}$$

$$\nabla \boldsymbol{v} = 0 \tag{5}$$

$$\nabla \boldsymbol{b} = 0 \tag{6}$$

$$\nabla x \boldsymbol{b} = 4\pi \boldsymbol{j} \tag{7}$$

$$\partial \boldsymbol{b}/\partial t = \nabla x(\boldsymbol{v} x \boldsymbol{B}) \tag{8}$$

$$v_r(R_1) = v_r(R_2) = 0. \tag{9}$$

Here $\boldsymbol{b}$, $\boldsymbol{v}$, and $\boldsymbol{j}$ are the perturbation values of the field, velocity, and current ($p$ is the pressure).

These equations separate. We must write $\boldsymbol{b}$ and $\boldsymbol{v}$ in the Lüst and Schlüter form (1954)

$$\boldsymbol{h} = (\boldsymbol{B}+\boldsymbol{b})/\sqrt{4\pi\varrho} = rTi_\Phi + \nabla x(rPi_\Phi) \tag{10}$$

$$\boldsymbol{v} = rV\boldsymbol{i}_\Phi + \nabla x(rU\boldsymbol{i}_\Phi), \tag{11}$$

use the equations given by Chandrasekhar for the functions $T$, $P$, $U$, and $V$ in 1956, separate them (now a question of algebra), and return to $b$ and $v$. After tedious but simple calculations one obtains

$$\{(\omega^2-\omega_0^2)^2(k^2-d/dr\, r^{-1}\, d/dr\, r)-2(B_\Phi B_z^{-2}\omega^2\omega_0^2 r^{-1})\, [\omega_0^2\omega^{-2}(2B_\Phi r^{-1}+ + rd/dr(B_\Phi r^{-1}))-rd/dr(B_\Phi r^{-1})]\} \begin{pmatrix} b_r \\ v_r \end{pmatrix} = 0 \tag{12}$$

$$r^{-1}d/dr\, r \begin{pmatrix} b_r \\ v_r \end{pmatrix} = -ik \begin{pmatrix} b_z \\ V_z \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} v_r \\ v_z \end{pmatrix} = \frac{-1}{\sqrt{4\pi\varrho}} \begin{pmatrix} b_r \\ b_z \end{pmatrix} \tag{14}$$

$$(\omega^2-\omega_0^2)b_\Phi = -i\omega^2 k^{-1}B_z^{-1}[(2B_\Phi r^{-1}+rd/dr(B_\Phi r^{-1}))\omega_0^2\omega^{-2}-rd/dr(B_\Phi r^{-1})]b_r \tag{15}$$

$$(\omega^2-\omega_0^2)v_\Phi = -2i\omega_0^2 B_\Phi k^{-1}B_z^{-1}r^{-1}v_r. \tag{16}$$

Here

$$\omega_0^2 = k^2B_z^2/4\pi\varrho = k^2u_a^2. \tag{17}$$

For a current flowing down the centre $B_\Phi = 2Ir^{-1}$ and we obtain all the relevant equations of the author's 1967 paper.

### 3. *The constant current case*

Before we move on to more general considerations we will solve the constant current case. Now (no inside cylinder)

$$R_1 = 0 \tag{18}$$

$$B_\Phi = 2IR_2^{-2}r = \beta r \tag{19}$$

and equation (12) becomes

$$(d^2/dr^2+r^{-1}d/dr+(\lambda^2-k^2)-r^{-2})v_r = 0 \tag{20}$$

$$\lambda^2 = 4\beta^2\omega_0^4/B_z^2(\omega^2-\omega_0^2)^2, \tag{21}$$

and the solution must be zero at $R_2$ and finite inside the cylinder. Therefore if $\lambda^2-k^2 > 0$ the only possible solutions are of the type

$$v_r = \text{const} \cdot J_1(j_{1m}R_2^{-1}r)e^{i(kz+\omega t)} \tag{22}$$

giving a dispersion relation:

$$\lambda_m^2 = j_{1m}^2R_2^{-2}+k^2 = 4\beta^2\omega_0^4/B_z^2(\omega_m^2-\omega_0^2)^2. \tag{23}$$

Here $j_{1m}$ is the $m$ th zero of the first Bessel function $J_1$. If $\lambda^2-k^2 \leqslant 0$ no non-singular solutions of both the equations and the boundary conditions exist.

The dispersion relation (23) can be written as

$$\omega_m^2/\omega_0^2-1 = \pm 4I/B_zR_2^2\sqrt{j_{1m}^2R_2^{-2}+k^2} \tag{24}$$

one branch of which can give imaginary values $\omega_m$. For the minus branch $\omega_m^{(-)}$ is pure imaginary if

$$4I > B_zR_2^2\sqrt{j_{1m}^2R_2^{-2}+k^2}. \tag{25}$$

This is possible when

$$4I > B_zR_2^2j_{1m}R_2^{-1}, \tag{26}$$

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which, in terms of field intensities, would be $B_\Phi(R_2)B_z^{-1} > j_{1m}$. When inequality (26) holds all modes for $k < k_{cr}$, where

$$k_{cr}^2 = 16I^2R_2^{-4}B_z^{-2} - j_{1m}^2R_2^{-2} \tag{27}$$

are unstable. Calculations along the lines outlined by Sturrock (1958) show this instability to be nonconvective. This should make it easy to detect experimentally. Dispersion relations for various values of $I/B_z$ are given in Figure 1. The dependence of the frequencies on the current is given in Figure 2. We see (as we do from equation (24)) that

$$\lim_{I\to 0} \omega_m^2(I) = \omega_0^2. \tag{28}$$

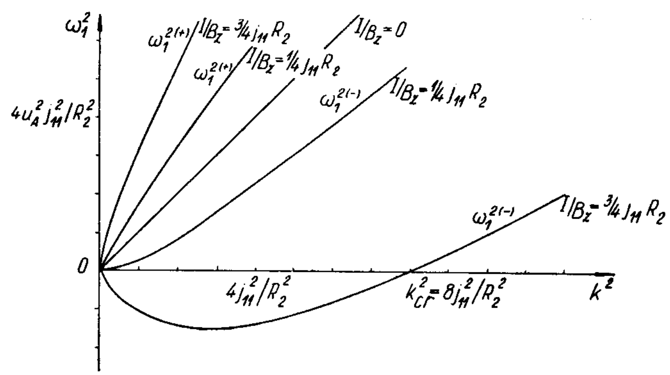


Fig. 1. Dispersion curves for constant current case, $m = 1$, and three values of the total current $I$. The region of negative $\omega_1^2$ is unstable

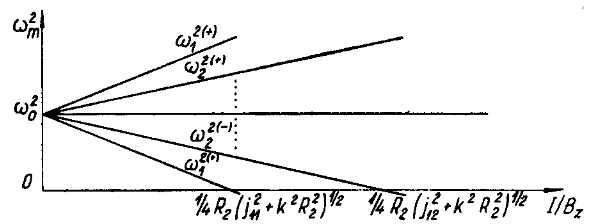


Fig. 2. Dependence of $\omega_m^{2(\pm)}$ on the current for constant current and given $k$. The square of the Alfvén frequency is $\omega_0^2$

The limit values of $v_r$, $v_\Phi$, and $v_z$ given by equations (22), (14) and (16) for $I \to 0$ are not the general solutions of the initial equations for $I = 0$. For a discussion of this fact see the author's 1967 paper. Figure 3 gives the division of the data into stable and unstable half planes.

When the cylinder is closed and of length $L$, $v_r$ becomes

$$v_r = \text{const} \cdot J_1(j_{1m}R_2^{-1}r)\sin(k_p z) \tag{29}$$

$$k_p = p\pi L^{-1}. \tag{30}$$

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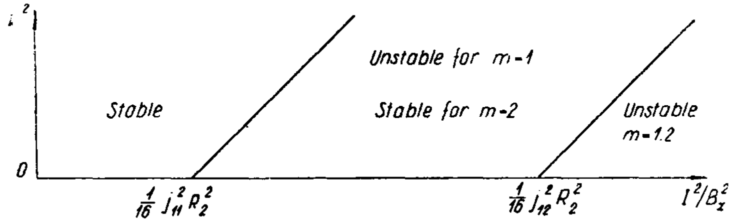


Fig. 3. Regions of stable and unstable solutions for constant current and first two modes

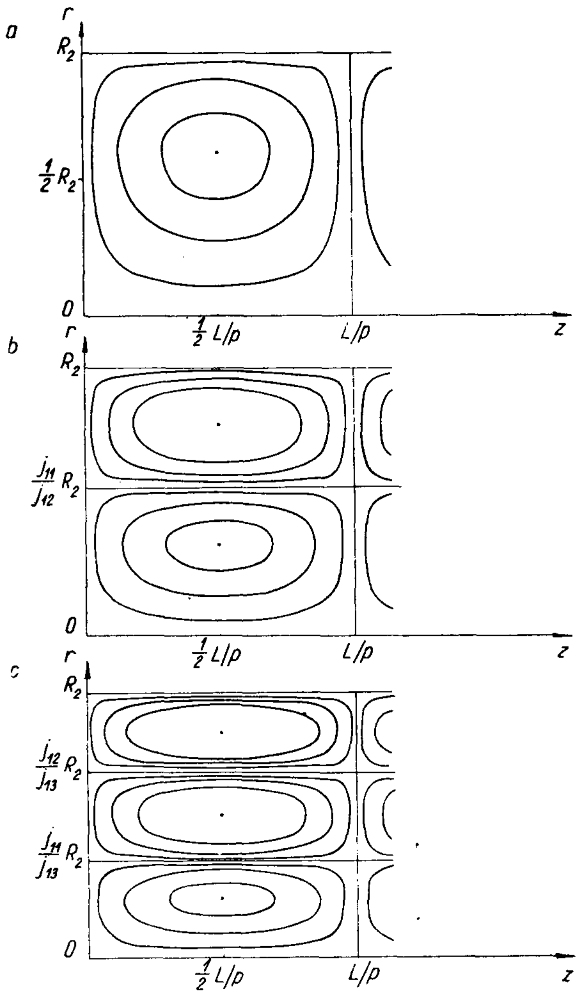


Fig. 4. The flow surfaces for constant current case and first three modes. The surfaces are obtained by rotating each drawing round the $z$ axis and repeating it $p$ times on the right. Drawings $a$, $b$, and $c$ correspond to eingenfrequencies for $m = 1, 2,$ and $3$

The flow surfaces for $m = 1, 2$, and 3 (and any $p$) can be visualised with the help of Figure 4. They are given by the rotation of the curves $rJ_1(j_{1m}R_2^{-1}r) = \text{const}$ round the $z$ axis. This follows from the fact that $v_r = \text{const} \cdot rU$ (see equation (11)) and

$$[\nabla \times (\psi i_\Phi)] \cdot \Delta(r\psi) = 0 \tag{31}$$

for $\psi$ independent of $\Phi$.

The $B_\Phi = \text{const}$ case can also be solved, yielding instability. The solutions are of the form $\alpha_1 I_x(kr) + \alpha_2 K_x(kr)$ and the dispersion relation is now

$$2B_\Phi^2 k^2 u_a^2(\omega_m^2 + k^2 u_\alpha^2)/B_z^2(\omega_m^2 - k^2 u_a^2)^2 = (x_m(k)+1)^2 \tag{32}$$

where $x_m(k)$ is the $m$ th root of

$$I_x(kR_1)\, K_x(kR_2) - I_x(kR_2)\, K_x(kR_1) = 0. \tag{33}$$

If we solve equation (32) we see that $\omega_m^2$ can be negative for $B_z(x_m+1)/B_\Phi < 2^{\frac{1}{2}}$.

## 4. *General criteria*

If we limit our attention to fields given by

$$B_\Phi = \beta r^n \quad (n \text{ real}) \tag{34}$$

we can find the values of $n$ for which the system can become unstable. From the author's 1967 paper we know that this cannot happen for $n = -1$ (we must remember that everything said here is limited to axially symmetric instabilities). We saw in Chapter 3 that the system can become unstable for $n = 0$ and $n = 1$. We will now show that this is true of all $n > -1$.

For fields given by equation (34), equation (12) becomes

$$(k^2 - d/dr\, r^{-1}\, d/dr\, r - \lambda^2 r^{2n-2}) \begin{pmatrix} b_r \\ v_r \end{pmatrix} = 0, \tag{35}$$

where

$$\lambda^2 = \frac{2\beta^2\omega_0^2[(n+1)\omega_0^2 - (n-1)\omega^2]}{B_z^2(\omega^2-\omega_0^2)^2}. \tag{36}$$

By integrating equation (35) multiplied by $rv_r$ or $rb_r$ on the left, between $R_1$ and $R_2$, and using equation (9), we see that $\lambda^2 > 0$. This will be true for all possible values of $\omega$. Therefore

$$\frac{n+1-(n-1)\omega^2\omega_0^{-2}}{(1-\omega^2\omega_0^{-2})^2} > 0. \tag{37}$$

This inequality will tell us when $\omega^2$ is positive, and when negative. If we write it as

$$n+1-(n-1)\,\omega^2\omega_0^{-2} = |\alpha|(1-\omega^2\omega_0^{-2})^2, \tag{38}$$

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where $\alpha$ is a real number, we can solve for $\omega^2\omega_0^{-2} = x$. We obtain

$$2x_1 = 2-(n-1)/|\alpha|+\Delta^{\frac{1}{2}} \tag{39}$$

$$2x_2 = 2-(n-1)/|\alpha|-\Delta^{\frac{1}{2}} \tag{40}$$

$$\Delta = (2-(n-1)/|\alpha|)^2-4(1-(n+1)/|\alpha|) = (n+1)^2/|\alpha|^2+8|\alpha|^{-1}. \tag{41}$$

We see from equation (41) that $\Delta > 0$, so $x_1$ and $x_2$ are real. The frequencies are real or pure imaginary, as was shown generally by Chandrasekhar in 1961. For $n = -1$ both $x_1$ and $x_2$ are positive and the system is stable, confirming the results of the author's 1967 paper. For $n < -1$ both $x_1$ and $x_2$ are positive, as $0 < \Delta < 2-(n-1)/|\alpha|$. Combining these two results we see that *the system is stable wtih respect to the axially symmetric instability when* $n \leqslant -1$.

A little algebra gives negative values of $x_2$, and hence instability, for the $n > -1$ case when

$$|\alpha| < n+1, \text{ or}^1 \tag{42}$$

$$B_z^2(R_2^{2n+2}-R_1^{2n+2})\lambda_m^2/8I^2 = B_z^2\lambda_m^2/2\beta^2 < n+1. \tag{43}$$

The last equation gives both equation (25) and the condition at the end of Chapter 3 for $n = 1$ and $n = 0$ respectively. We see that increasing the entire current beyond $I_{cr}$, where

$$I_{cr} = [B_z^2(R_2^{2n+2}-R_1^{2n+2})\,\lambda_m^2/8(n+1)]^{\frac{1}{2}}, \tag{44}$$

will destabilize the system for a given mode. To evaluate $I_{cr}$ and $k_{cr}$ we need an evaluation of $\lambda_m$.

The transformation $w = rv_r$ gives equation (35) a simpler form. Now $w(R_1) = w(R_2) = 0$ and

$$d/dr\, r^{-1}dw/dr-k^2r^{-1}w+\lambda^2r^{2n-3}w = 0, \tag{45}$$

$$\lambda^2 = \frac{\int\limits_{R_1}^{R_2}(dw/dr)^2r^{-1}dr+\int\limits_{R_1}^{R_2}k^2w^2r^{-1}dr}{\int\limits_{R_1}^{R_2}w^2r^{2n-3}dr}. \tag{46}$$

Any function substituted for $w(r)$ and satisfying the boundary conditions will give an evaluation of $\lambda_1$ from above. If $\tilde{w}(r)$ is any function that vanishes at $R_1$ and $R_2$, and $\tilde{\lambda}_1$ is defined by equation (46) with $\tilde{w}(r)$ in place of $w(r)$, then

$$\lambda_1 \leqslant \tilde{\lambda}_1 \tag{47}$$

and we have an evaluation of $I_{cr}$ from above.

[1] $\lambda_m$ is the $m$ th eigenvalue of equation (25).

If

$$F_1 = \int_{R_1}^{R_2} (dw/dr)^2 r^{-1} dr$$

$$F_2 = \int_{R_1}^{R_2} w^2 r^{-1} dr$$

$$F_3 = \int_{R_1}^{R_2} w^2 r^{2n-3} dr \tag{48}$$

then

$$\frac{1}{2} B_z^2 \beta^{-2}(F_1/F_3 + k^2 F_2/F_3) < n+1 \tag{49}$$

will be a sufficient condition for instability of the first mode, but not of the higher modes as $\lambda_1^2 \leqslant \lambda_m^2$ for all $m \geqslant 1$. The lowest mode becomes unstable first.

Equation (49) can be used to evaluate $I_{cr}$, as it will give an estimate from above. It also shows that there is a critical value of $k$ for each mode.

## 5. *Summary*

We see that for azimuthal magnetic fields of the type $\beta r^n$ instabilities of a perfectly conducting fluid are possible when $n > -1$, if the total current is greater than a critical value, and the wave length also exceeds a critical value. All these critical quantities can be given approximately or, in two cases, exactly. In one case the instability was shown to be absolute.

It has also been established that $\omega^2$ must be real, as expected, and that the frequency is split up into a spectrum $\omega_m$, where $\lim_{I\to 0} \omega_m^2 = \omega_0^2$. This is a generalization of the result of the author's 1967 paper, treating the $n = -1$ case in some detail.

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