COVARIANT FORMULATION OF THE CAUCHY PROBLEM IN GENERALIZED ELECTRODYNAMICS AND GENERAL RELATIVITY

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In memory of the great Polish physicist, Leopold Infeld

The Cauchy problem in a manifold is formulated geometrically as the problem of the continuation of certain field variables off a rigged hypersurface of the manifold, given certain initial data on that hypersurface, and a system of covariant partial differential equations obeyed by the field variables. The Lie derivative is shown to lend itself naturally to such a formulation. The Cauchy problem for generalized electrodynamics and spacelike initial hypersurfaces in general relativity is formulated and discussed from this point of view. It is shown that without the constitutive equations the propagation characteristics of Maxwell's equations are undefined. A Newtonian form for the field equations of general relativity is given, and a Hamiltonian form using Lie derivatives is developed from the Lagrangian for the field equations. All calculations are explicitly covariant, and the geometrical interpretation of the results thereby becomes obvious.

I. Introduction

In field theories which can be formulated in terms of hyperbolic partial differential equations, the Cauchy or initial-value problem for spacelike hypersurfaces is of some importance[1]. It is a well-posed problem mathematically, and its solution is of physical interest. For example, it enables solutions to the field equations to be classified in terms of the minimal independent initial data needed to specify a solution to the Cauchy problem.

In most field theories (including all special-relativistic ones), it is possible to give a covariant formulation of the Cauchy problem using the covariant derivative with respect to the background metric (*e.g.*, Minkowski metric tensor) for the space-time manifold. In general relativity, however, where the metric tensor itself is always one set of field variables to be determined by the field equations (and, indeed, for the empty-space Einstein field

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equations is the only set of field variables involved) this is not possible. One might introduce a second background metric (actually, an affinity is all that would be needed), given *a priori*; and introduced just so that covariant derivatives with respect to it could be defined. But this seems highly artificial; both physically, since there is no motivation for the introduction of a second metric or affinity; and mathematically, since it introduces unnecessary geometrical elements into the formulation of the problem.

For this reason, the Cauchy problem in general relativity is usually formulated in a non-covariant way, using a particular coordinate system, and partial derivatives with respect to that system (indeed, this is how the Cauchy problem is usually formulated even in special relativistic theories; but it can be reformulated using covariant derivatives quite easily). This naturally leads one to suspect that the coordinate system has been tacitly adapted to some geometrical structure related to the Cauchy problem; and one must then attempt to disengage this structure from the non-covariant formalism.

We shall pursue the opposite path here and show that the Cauchy problem in a space-time manifold involves a definite geometrical structure; and may be covariantly formulated from the start when this structure is explicitly introduced. The usual formulations will then follow naturally, when a coordinate system is adapted to this structure. The formulation of the problem in this geometrical way gives more intuitive insight into the nature of the problem and the structure of the field equations.

We shall first discuss the general structure of the Cauchy problem for a covariant field theory in a space-time manifold, and how it naturally leads to the use of the Lie derivative. We shall next discuss this technique in the case of generalized electrodynamics. By this, we mean that part of electrodynamics (Maxwell's Equations) which is independent of the constitutive relations linking $\vec{D}$ and $\vec{H}$ with $\vec{E}$ and $\vec{B}$. Then, we shall apply this approach to the Cauchy problem in general relativity, using a geodesic normal field to propagate off an initial hypersurface, yielding what we shall call a Newtonian approach to the problem, since it involves the use of the field equations to directly compute second Lie derivatives or "field accelerations". Next we shall show how the usual Lagrangian formalism of general relativity may be cast into a form which leads to the same form of the equations of motion; and how a Hamiltonian formalism using Lie derivatives may thereby be set up. Finally, we shall indicate briefly how the problem may be formulated for an arbitrary vector field, and discuss the structure of the constraint equations in this case[4].

The study of the Cauchy problem in general relativity goes back to the early days of the development of the theory[2]. In recent years, there has been renewed interest in the problem in connection with the effort to isolate and study the true degrees of freedom of the gravitational field, or "observables" as they are often called[3].

Many of the results included in this paper may already be found in the literature in the form of non-covariant equations in particular coordinate systems, whose geometrical significance may or may not have been noted. We believe it worth-while to present these results in a unified, systematic way, developing them in a covariant form which automatically implies their geometric interpretation.

Throughout this paper, we shall consider only the formal aspects of the solution to the Cauchy problem, assuming analyticity of the solutions, so that series expansion techniques

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may be used. Actual proofs of the existence of a local solution to the problem for the field equations of general relativity for non-analytic initial data; the well-posed nature of the problem for a system of hyperbolic-normal partial differential equations (such as the field equations of general relativity); domains of influence and similar questions are, of course, best treated by standard methods of the theory of partial differential equations in particular coordinate systems adapted to the problem[5].

It is very important, in principle, to consider non-analytic and even non-continuous solutions to hyperbolic partial differential equations, as in the study of the propagation of shock waves, for example. However, once one knows that such solutions are possible, the restriction to analytic solutions is not so important in practice. A finite discontinuity may always be approximated by a continuous function varying rapidly in the neighborhood of the discontinuity. And, if continuous initial data is given over any bounded region of the initial hypersurface, it only determines a solution within its domain of influence, which will be finite. But any continuous function may be approximated as closely as desired in any bounded domain by analytic functions, which may also be chosen to approximate its derivatives, if continuous, up to any finite order[6].

For the benefit of those not familiar with some of the concepts from differential geometry needed in this paper, such as the Lie derivative, a brief discussion is included in the Appendix[7].

II. Nature of the Cauchy problem

Looking at the Cauchy problem for a physical field geometrically, it is seen to involve two questions: *1.* the initial value problem: determination of the data about some field or fields, which must be given on an initial hypersurface of the space-time manifold in order to determine a unique solution to the field equations obeyed by the fields throughout some region of space-time including the initial hypersurface; *2.* the problem of evolution: the determination of the evolution of the field off the initial hypersurface. If the fields obey generally-covariant equations, or otherwise are invariant under some gauge group involving one or more arbitrary functions, we shall have constraint equations; *i.e.*, relations among the field variables and their derivatives on the initial hypersurface which limit the possible choice of initial values for all the field variables in some way. In addition, the gauge freedom implies that the propagation of *all* the field variables cannot be uniquely determined as to functional form by any amount of data on the initial hypersurface, due to the gauge freedom of transformations off the initial hypersurface. These functionally different solutions must be equivalent modulo coordinate and gauge transformations, of course[8]. The field equations will then split into two sets: the constraint equations on the initial hypersurface, and the propagation equations, yielding the values of certain component of the field off the initial surface.

We are led, in studying the propagation off the initial hypersurface, to the idea of a family of hypersurfaces, in which our initial hypersurface is embedded; and of the propagation of field values from surface to surface. In a Galilei-invariant theory, or in a Lorentz-invariant theory in which we use a space-like hyperplane as the initial surface, it is quite natural to take parallel space-like hypersurfaces in space-time (*i.e.*, the surfaces: absolute

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or inercial-system time equals constant). It is equally natural to identify those points on each hypersurface which have the same Cartesian spatial coordinates. The formulation of the problem in such terms involves only ordinary derivatives with respect to these preferred coordinates; but it is trivially obvious how to generalize this to a formulation in terms of arbitrary coordinates and covariant derivatives. The simplicity of these special constructions may cause one to lose sight of the fact not only must a family of hypersurfaces be chosen (quite arbitrarily), but that a correlation between points on the hypersurfaces must be set up if we want to formulate a solution to the evolution problem starting from the initial data. The specification of a family of hypersurfaces containing an initial hypersurface, as well as a correlation between points on the initial hypersurface and those on each hypersurface of the family, is equivalent to the specification of the initial hypersurface together with some contravariant vector field transvecting it (*i.e.*, such that $V^{\mu}\Phi,_{\mu} \neq 0$, where V^{μ} is the vector field, and $\Phi = 0$ is the equation of the initial hypersurface). Technically, the initial data must be given on a hypersurface rigged with a contravariant vector field[9]. Then, the dragging along of the points of the initial hypersurface with the vector field (more correctly, with the one-parameter group of point transformations generated by the vector field) gives rise to just the desired family of hypersurfaces, with a correlation between the points of each hypersurface which lie on a common trajectory of the group of transformations. Now, given some set of sufficient initial data, we can formulate the problem of evolution covariantly by saying that we are going to compare the actual values of the field at each point with the dragged-along values of the initial data at that point. But this means that, in the infinitesimal limit, we are computing the Lie derivatives of the initial data with respect to the vector field generating the group of transformations. Or putting it the other way around, given the appropriate field components and certain of their Lie derivatives with respect to the vector field (initial data), the field equations must allow us to compute all higher Lie derivatives of the initial data fields with respect to an arbitrary contravariant vector field, if we are to be able to compute the fields at all points; *i.e.*, if the data really do determine a solution to the field equations.

Thus, the Cauchy problem for any set of field equations is most naturally described geometrically in terms of Lie derivatives. This is not surprising if we remember that the Lie derivative is the covariant generalization of the ordinary partial derivative with respect to *one* coordinate variable[10].

The concept of Lie derivative involves only the introduction of a contravariant vector field into a bare manifold; thus, our formulation of the Cauchy problem involves no metrical (or affine) concepts. We have used the ideas of hypersurfaces and contravariant vector fields, in addition to the field variables (with whatever transformation properties they may have) and nothing more. Thus, we can formulate the Cauchy problem for any field of geometrical objects regardless of whether a metric tensor or other background fields (*i.e.*, fields unaffected by the field equations) are present. At most, they will enter as "parameter fields" into our formulation, perhaps occurring in the field equations and thus in our solution to the problem.

It is relatively simple to formulate the Cauchy problem for any of the usual special-relativistic fields in this way: for the Klein–Gordon field, the usual Maxwell equations in Minkowski space-time, *etc.*, We shall not bother here with the details of such a reformulation.

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However, the formulation of the problem when no metric is given *a priori* is rather more interesting. Here two cases come to mind: generalized gauge-invariant electrodynamics without specification of particular constitutive equations[11] (either in a material medium or in empty space; this latter case includes all Born–Infeld type theories[12]; and the field equations of general relativity. We shall discuss the case of generalized electrodynamics in the next section and then turn to the empty-space Einstein equations in the rest of the paper.

Before going on to discuss the case of generalized electrodynamics in a bare manifold, we mention here a particular choice of the relationship between the vector field v^μ and the family of hypersurfaces $\Phi = \text{const}$, which can always be fulfilled locally in any manifold; and which is often most useful in setting up the Cauchy problem. We can always so arrange the relationship as to satisfy the following condition:

$$v^\mu \Phi,_\mu = 1. \tag{2.1}$$

If $v^\mu \Phi,_\mu \equiv 1$ in some region, then since $\underset{v}{\pounds}$ and ∂_μ commute, it follows that

$$\underset{v}{\pounds}\Phi,_\mu = \partial_\mu \underset{v}{\pounds}\Phi = \partial_\mu(v^\varkappa \Phi,_\varkappa) = 0, \tag{2.2}$$

where $\Phi,_\mu$ is the covariant normal to the surface. Condition (2.1) takes the place on a rigged hypersurface of having a unit contravariant normal vector in a metric space; and Eq. (2.2) materially simplifies the calculation of Lie derivatives of projections onto the "normal" direction. In a metric space, we can always arrange to satisfy these intrinsically non-metric properties by indentifying v^μ with the unit geodesic normal field to our hypersurface, as we shall later discuss in more detail when we come to consider general relativity.

III. Generalized electrodynamics

As first pointed out in the early 1920's,[13] the general structure of Maxwell's equations can be formulated in a bare manifold since they involve no metric or affine concepts. It is only the constitutive equations relating $\vec{D}$ and $\vec{H}$ to $\vec{E}$ and $\vec{B}$ which introduce relationships which involve a metric. By suitable choice of these constitutive equations, we get Maxwell's linear electrodynamics for empty space-time, non-linear Born-Infeld type of equations[12] for empty space time (classical "polarization of the vacuum"); or Maxwell's equations for any medium.[11] We shall call the bare Maxwelle equations without any particular choice of constitutive equations "generalized electrodynamics". As is well-known, we need two second rank antisymmetric tensor densities $\tilde{F}^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ for generalized electrodynamics; these correspond to the usual three-dimensional $\vec{E}$ and $\vec{B}$, and $\vec{D}$ and $\vec{H}$, respectively.[12] Then the generalized Maxwell equations take the form:

$$\partial_\mu(\tilde{F}^{\mu\nu}) = 0, \tag{3.1}$$

$$\partial_\mu(\tilde{G}^{\mu\nu}) = \tilde{j}^\nu, \tag{3.2}$$

where $\tilde{j}^\nu$ is a current density for the sources of the $\tilde{G}^{\mu\nu}$ fields. Notice these are tensorial equations involving no metrical concepts at all. As mentioned above, the choice of a particular

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electrodynamic theory arises only when a relationship between $\tilde{F}^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ is specified: $\tilde{G}^{\mu\nu} = \tilde{G}^{\mu\nu}[\tilde{F}^{\varkappa\lambda}]$. These relations are called the constitutive equations of the theory. The non-metrical conservation equation $\partial v \tilde{j}^{\nu} = 0$ follows as a consequence of (3.2). We shall first discuss the Lie derivative approach to the Cauchy problem for equations (3.1) and (3.2). We use a vector field and family of hypersurfaces so adapted to our initial rigged hypersurface that conditions (2.1) hold (as well as all their Lie derivatives, of course). For brevity, we shall write $v_\mu = \Phi,_\mu$, but it should be emphasized that this is not the usual metric correlation between a contravariant and covariant vector (indeed, the conditions (2.1) do not even uniquely define v_μ in terms of v^μ). We break up the tensor densities $\tilde{F}^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ with respect to the "normal" and "hypersurface" directions (in this generalized, non-metrical sense of "normal" and "hypersurface"), using $C^\nu_\mu = v^\nu v_\mu$ as the projection operator for normal components and $B^\nu_\mu = \delta^\nu_\mu - v^\nu v_\mu$ as the projection operator for hypersurface components. When applied to an antisymmetric tensor density like $\tilde{G}^{\mu\nu}$, we see that there will be a normal-surface component of $\tilde{G}^{\mu\nu}$ defined by

$$\tilde{D}^\mu = \tilde{G}^{\alpha\beta} B^\mu_\alpha v_\beta, \quad \tilde{D}^\mu v_\mu = 0 \qquad (3.3)$$
$$= \tilde{G}^{\mu\beta} v_\beta$$

and an antisymmetric surface-surface component, defined by

$$\tilde{H}^{\mu\nu} = \tilde{G}^{\alpha\beta} B^{\mu\nu}_{\alpha\beta}, \quad \tilde{H}^{\mu\nu} v_\nu = 0, \qquad (3.4)$$

where $B^{\mu\nu}_{\alpha\beta}$ (and all longer strings of indices attached to B or C) is shorthand for $B^\mu_\alpha B^\nu_\beta$. Note there are no normal-normal components because of the antisymmetry of $\tilde{G}^{\alpha\beta}$, so $\tilde{D}^\mu$ and $\tilde{H}^{\mu\nu}$ exhaust the content of $\tilde{G}^{\mu\nu}$. We have chosen the abbreviations $\tilde{D}^\mu$ and $\tilde{H}^{\mu\nu}$ because, as we shall see, these densities play the role, with respect to our breakup on the hypersurface, that $\vec{D}$ and $\vec{H}$ play in the usual breakup with respect to an inertial frame. We shall also break up the current density vector:

$$\tilde{\varrho} = \tilde{j}^\mu v_\mu, \quad \tilde{s}^\mu = \tilde{j}^\mu - \tilde{\varrho} v^\mu = \tilde{j}^\nu B^\mu_\nu, \quad \tilde{s}^\mu v_\mu = 0. \qquad (3.5)$$

Then taking the normal component of (3.2) and remembering that $\partial_\mu v_\nu = \partial_\nu v_\mu$, (since v_μ is a gradient) we see that

$$\partial_\mu (\tilde{D}^\mu) = \tilde{\varrho}. \qquad (3.2a)$$

Since $\tilde{D}^\mu$ is a hypersurface vector density, this must be regarded as a constraint on the choice of $\tilde{D}^\mu$ on the initial hypersurface. Indeed, it is just the generalized form of the usual div $\vec{D} = \varrho$ constraint. We should then expect the remainder of the field equations (3.2) to determine the evolution of $\tilde{D}^\mu$ off the initial hypersurface. Indeed, if we calculate $\underset{v}{\pounds}\, \tilde{D}^\mu$ using (3.2), we find that

$$\underset{x}{\pounds}\, \tilde{D}^\mu = \tilde{s}^\mu + \partial_\nu (\tilde{H}^{\nu\mu}), \qquad (3.2b)$$

which is the generalization of $(\partial \vec{D}/\partial t) = \vec{s} + \text{curl}\, \vec{H}$. Eq. (3.2) thus restricts the initial choice of $\tilde{D}^\mu$ by (3.2a), and then allows us to pick $\tilde{H}^{\mu\nu}$ and determine, via (3.2b), $\underset{v}{\pounds}\, \tilde{D}^\mu$. The data needed to specify a solution to (3.2) are then $\tilde{H}^{\mu\nu}$ and all its Lie derivatives on the initial hypersurface (*i.e.*, $\tilde{H}^{\mu\nu}$ throughout a region of space-time), the values of $\tilde{\varrho}$ and $\tilde{s}^{\mu}$; and the initial value of $\tilde{D}^{\mu}$ on the hypersurface subject to the constraint (3.2a). $\tilde{s}^{\mu}$ may be given throughout all space-time. The conservation equation then determines $\underset{v}{\pounds}\,\tilde{\varrho}$ once $\tilde{\varrho}$ is given on an initial hypersurface. A similar analysis of Eq. (3.1) may also be carried out. If we define $\tilde{E}^{\mu\nu}$ and $\tilde{B}^{\mu}$ by

$$\tilde{E}^{\mu\nu} = \tilde{F}^{\alpha\beta}B^{\mu\nu}_{\alpha\beta}, \qquad \tilde{B}^{\mu} = \tilde{F}^{\mu\nu}v_{\nu}, \tag{3.6}$$

then (3.1) breaks up into

$$\partial_{\mu}(\tilde{B}^{\mu}) = 0, \tag{3.1a}$$

the analogue of div $\vec{B} = 0$;
and

$$\underset{v}{\pounds}\,\tilde{B}^{\mu} = \partial_{\nu}(\tilde{E}^{\mu\nu}) = -\partial_{\nu}(\tilde{E}^{\nu\mu}), \tag{3.1b}$$

the analogue of $\dfrac{\partial\vec{B}}{\partial t} = -\text{curl}\,\vec{E}$.

Again, we may give $\tilde{B}^{\mu}$ on the initial hypersurface, subject to the constraint (3.1a), and $\tilde{E}^{\mu\nu}$ and all its Lie derivatives on the initial hypersurface, which will determine the Lie derivatives of $\tilde{B}^{\mu}$ via (3.1b). Needless to say, $\tilde{E}^{\mu\nu}$ and $\tilde{B}^{\mu}$ are the analogues of the usual $\vec{E}$ and $\vec{B}$.

It is not surprising that the fields $\tilde{E}^{\mu\nu}$ and $\tilde{H}^{\mu\nu}$ may so far be specified through all space-time; for it is just the constitutive equations, which we have ignored, which limit this freedom. In the usual Maxwell electrodynamics, for example, they link $\tilde{E}^{\mu\nu}$ to $\tilde{D}^{\mu}$, and $\tilde{H}^{\mu\nu}$ to $\tilde{B}^{\mu}$ so that only the initial values of $\tilde{E}^{\mu\nu}$ and $\tilde{H}^{\mu\nu}$ would be freely specifiable, all higher Lie derivatives being determined by the constitutive equations. Clearly, various constitutive assumptions will give rise to different Cauchy problems in electrodynamics; and the demand for a well-posed Cauchy problem may be used to limit the possible form of the constitutive equations. The lack of any inherent metric content in generalized electrodynamics is related to the fact that the Cauchy problem that arises for an arbitrary choice of the relationship between $\tilde{F}^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ does not necessarily lead to propagation along characteristics which coincide with the usual light cone, even though the resulting theory may be formally Lorentz invariant. This happens for certain Born–Infeld types of non-linear vacuum electrodynamic theory or, indeed, even in electrodynamics for a hypothetical medium with a linear constitutive relation between $\tilde{G}^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ satisfying the necessary criteria (discussed by Post[11] for example); but not involving the Minkowski metric explicitly. Here, the structure of the theory determines its own characteristic conoid at a point, which may even lie outside the local Minkowski light cone at that point. Indeed, if the constitutive equations should not involve the metric tensor at all, the equations would be Lorentz–invariant only in the sense that the Lorentz group is a sub-group of the much wider invariance group of general coordinate transformations. These considerations indicate the need for great care in distinguishing between formally Lorentz-invariant theories and physically Lorentz-invariant theories. We hope to return to this question in a separate paper.[15]

IV. General Relativity-Newtonian approach

We shall proceed in analogy with the last section by breaking up $g_{\mu\nu}$, the symmetric second rank tensor representing the gravitational field potentials, with respect to the normal and orthogonal projections onto v^μ, again assuming $v^\mu v_\mu = 1$, which simplifies the analysis considerably. There are now three types of projections: the transverse-transverse part $g^{tt}_{\mu\nu} = g_{\varkappa\lambda}B^{\varkappa\lambda}_{\mu\nu}$; the transverse-normal part $g^{tn}_{\varkappa} = g_{\mu\nu}B^{\mu}_{\varkappa}v^\nu$; and the normal-normal part $g^{nn} = g_{\mu\nu}v^\mu v^\nu$. When we have once built up our metric, there will have to be some metrical relation between v_μ and v^μ, which in general may involve four arbitrary functions:

$$v_\mu = a(g_{\mu\nu}v^\nu)+b_\mu,\ b_\mu v^\mu = 0. \tag{4.1}$$

These four functions really are arbitrary, and their evolution is unrestricted by the field equations. They determine the metric properties of the dragged along family of hypersurfaces. If we make the simplest possible assumption ($b_\mu = 0$, $a = 1$), then $v_\mu = g_{\mu\nu}v^\nu$, and it follows that g^{tn}_μ must vanish:

$$g^{tn}_\varkappa = (g_{\mu\nu}v^\nu)B^\mu_\varkappa = v_\mu B^\mu_\varkappa = 0; \tag{4.2}$$

and that g^{nn} must equal one:

$$g^{nn} = g_{\mu\nu}v^\mu v^\nu = v_\nu v^\nu = 1. \tag{4.3}$$

In a coordinate system adapted to the vectors v_μ and v^ν, *i.e.*, in which $v_\mu = \delta^0_\mu$ and $v^\mu = \delta^\mu_0$, we can easily see what this means: For then

$$g^{tt}_{\mu\nu} \stackrel{*}{=} g_{mn}\delta^m_\mu\delta^n_\nu,\ g^{tn}_\mu \stackrel{*}{=} g^m_{om}\delta_\mu,\ g^{nn} \stackrel{*}{=} g_{00}; \tag{4.4}$$

so that our condition $v_\mu = g_{\mu\nu}v^\nu$ makes $g_{o\mu} \stackrel{*}{=} \delta^o_\mu$-the so-called geodesic normal coordinates; *i.e.*, we have so constructed the metric as to make v^μ a geodesic normal field. This is one reason why the Cauchy problem works out most simply in general relativity with the choice of a geodesic normal field. We shall work with this choice throughout most of this paper. There is no reason, however, why we need be restricted to this choice, and indeed the choice of a and b_μ in Eq. (4.1) above is quite arbitrary, reflecting the fact that we can so build the metric as the give our vector field v^μ arbitrary metrical properties. Reference [18] works out some of the general consequences of such an arbitrary relationship; but in this paper we shall only look at the non-geodesic normal case in connection with the constraint equations in Section VI. If the background metric is given in a theory, the metric properties of a given vector field are prescribed, once the field is specified, of course. However, in general relativity, where the field equations precisely serve to determine the metric field, the metric properties of a vector field (given arbitrarily as a function of the coordinates in some coordinate system, say), are only determined after the metric is determined. From the point of view of the Cauchy problem in general relativity, this means that we may pick an arbitrary contravariant vector field in the manifold and build up the metric on the family of hypersurfaces it generates in such a way as to *give* the vector field any desired metric properties. In particular, we may make an arbitrary vector field a geodesic normal field with respect to the initial hypersurface by suitably building up the metric field off the initial hypersurface.

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The family of hypersurfaces that results from dragging the initial hypersurface along a geodesic normal vector field is the family of hypersurfaces geodesically parallel to the initial one (about the only preferred family of surfaces in which an arbitrary hypersurface in an arbitrary Riemann space can always be embedded); and points on the various hypersurfaces lying on the same geodesic are thereby correlated. We shall always use the notation n^μ for the unit field of tangents to the geodesics normal to the given initial hypersurface.

For simplicity we shall analyze the Cauchy problem by assuming that a solution to the field equations is given to us; and analyze the solution with respect to an initial hypersurface and the geodesic normal field to that hypersurface. It will then be seen, from our final results, that we could have reversed our reasoning and started with the initial data on a hypersurface of a bare manifold, proceeding to build up the metric structure in such a way that our vector field would indeed end up as the geodesic normal field to the initial hypersurface.

We now proceed to break up the metric $g_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\varkappa\lambda}$ in the normal and hypersurface directions with respect to the initial hypersurface. To project tensors with respect to their components normal to and in a given hypersurface, we again introduce the two projection operators $C^\mu_\nu = n^\mu n_\nu$ and $B^\mu_\nu = \delta^\mu_\nu - n^\mu n_\nu$, where n^μ is now the unit normal to the surface. When applied to any tensorial index, C^μ_ν and B^μ_ν project out the normal and orthogonal parts, respectively.

As seen above, when we project the metric tensor $g_{\mu\nu}$ we get only a normal-normal part, which is $n_\mu n_\nu$ itself, and an orthogonal-orthogonal part, which is $g_{\mu\nu} - n_\mu n_\nu = {}'g_{\mu\nu}$. Now ${}'g_{\mu\nu}$ is just the first fundamental form or inner metric of the initial hypersurface.[16] For a geodesic normal field, the Lie derivative of n_μ with respect to n^ν vanishes (this is why the geodesic normal field introduces so much simplification into the formulae), so that the normal-normal part of the field on any hypersurface is just the dragged along field itself. We thus reduce the evolution problem to consideration of the evolution of ${}'g_{\mu\nu}$. The Lie derivative of ${}'g_{\mu\nu}$ is easily shown to be $-2h_{\mu\nu}$, where $h_{\mu\nu}$ is the second fundamental form of the surface.[17] Thus, giving ${}'g_{\mu\nu}$ and $\pounds\, {}'g_{\mu\nu}$ on a hypersurface is geometrically equivalent to giving the first and second fundamental forms of the hypersurface. It is a simple and direct matter now to calculate the second Lie derivative of ${}'g_{\mu\nu}$ on the initial hypersurface; and then use the field equations to evaluate this in terms of ${}'g_{\mu\nu}$ and its first Lie derivative on the surface. That is why we call this method Newtonian; it is formally much like writing $m\vec{a} = \vec{F}$, and then using the force law to evaluate $\vec{F}$ as a function of $\vec{x}$ and $\vec{v}$. A short calculation (using the properties of a geodesic normal field) shows that

$$\underset{n}{\pounds^2}\, {}'g_{\mu\nu} = 2(h^\lambda_\mu h_{\lambda\nu} - B^{\alpha\beta}_{\mu\nu} R_{\varkappa\alpha\beta\lambda} n^\varkappa n^\lambda), \tag{4.5}$$

(where $B^{\alpha\beta}_{\mu\nu}$ is again just the abbreviation for $B^\alpha_\mu B^\beta_\nu$). We are thus led to examine the projections of the Riemann tensor in normal and orthogonal directions. Here there are three non-vanishing types of projections: four times orthogonally onto the surface; three times orthogonally onto the surface and once onto the normal direction; and twice orthogonally and twice normally (onto different index pairs). The first two projections are completely determined

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by the first and second fundamental forms on the hypersurface. Indeed this is just the content of the Gauss–Codazzi equations:[7], [18]

$$B^{\varkappa\lambda\mu\nu}_{\alpha\beta\varrho\delta} R_{\varkappa\lambda\mu\nu} = {}'R_{\alpha\beta\varrho\delta} + h_{\alpha\varrho} h_{\beta\delta} - h_{\alpha\delta} h_{\beta\varrho}, \tag{4.6}$$

$$B^{\varkappa\lambda\mu}_{\alpha\beta\varrho} n^{\nu} R_{\varkappa\lambda\mu\nu} = {}'\nabla_{\alpha} h_{\beta\varrho} - {}'\nabla_{\beta} h_{\alpha\varrho}. \tag{4.7}$$

These equations are thus seen to relate the metric and second fundamental form on a hypersurface to the components of the Riemann tensor of the four-space, projected four times and three times, respectively, onto the surface. When we come to apply the field equations $G_{\mu\nu} = 0$, they will lead to constraints on the first and second fundamental forms on the initial hypersurfaces.

The first and second fundamental forms are equivalent to the metric on the initial hypersurface and the first neighboring geodesically parallel hypersurface, as we see at once from the fact that the second fundamental form is the first Lie derivative of the first fundamental form. From Eq. (4.5), we see that calculation of the second Lie derivative, which is equivalent to the metric on the second geodesically parallel neighboring hypersurface, requires just the components of the Riemann tensor projected twice on the surface and twice normally. With no field equations restricting the metric, these would, of course, be freely specifiable; it is just the surface projection of Einstein field equations that serve to fix these components. Indeed, it is easily shown by direct computation that

$$B^{\mu\nu}_{\varkappa\lambda} R_{\mu\nu} = {}'R_{\varkappa\lambda} + h^{\varrho}_{\varkappa} h_{\varrho\lambda} - h^{\varrho}_{\varkappa} h_{\varrho\lambda} + B^{\mu\nu}_{\varkappa\lambda} n^{\varrho} n^{\delta} R_{\varrho\mu\nu\delta}, \tag{4.8}$$

so that

$$\underset{n}{\pounds}^{2}\, {}'g_{\mu\nu} = 2({}'R_{\mu\nu} + 2h^{\varrho}_{\mu} h_{\varrho\nu} - h^{\varrho}_{\varrho} h_{\mu\nu}) - B^{\alpha\beta}_{\mu\nu} R_{\alpha\beta}. \tag{4.9}$$

Thus, if we assume that $B^{\alpha\beta}_{\mu\nu} R_{\alpha\beta} = 0$, it follows that

$$\underset{n}{\pounds}^{2}\, {}'g_{\mu\nu} = 2({}'R_{\mu\nu} + 2h^{\varrho}_{\mu} h_{\varrho\nu} - h^{\varrho}_{\varrho} h_{\mu\nu}), \tag{4.10}$$

which are just the equations of evolution for ${}'g_{\mu\nu}$, since they express the second Lie derivatives of ${}'g_{\mu\nu}$ in terms of the ${}'g_{\mu\nu}$ and their first Lie derivatives. Thus the field equations (strictly the components $B^{\mu\nu}_{\alpha\beta} R_{\mu\nu}$ of the field equations) serve to determine just the needed inner metric on the second geodesically parallel neighboring hypersurface.

It is clear that this was the crucial step in the procedure; all higher Lie derivatives may be found by iteration, in terms of lower order Lie derivatives; for an analytic solution to the field equations, this allows us to drag the ${}'g_{\mu\nu}$ field on the initial hypersurface $t = 0$ along a parameter distance $\boldsymbol{t}$, to an arbitrary geodesically parallel hypersurface:

$${}'g_{\mu\nu}[t] = e^{t\underset{n}{\pounds}}\, {}'g_{\mu\nu}[0], \qquad (4.11$$

in a fairly obvious symbolic notation. Then $g_{\mu\nu}(t)$ on that hypersurface equals ${}'g_{\mu\nu}(t) + n_{\mu}(t) n_{\nu}(t)$, where $n_{\mu}(t)$ is the dragged-along normal field since $\underset{n}{\pounds} n_{\mu} = 0$. So $g_{\mu\nu}(t) = {}'g_{\mu\nu}(t) + n_{\mu} n_{\nu}$.[19]

So far, we have not considered the normal-normal and normal-hypersurface components of the field equations, which give rise to the constraints on the initial data. Using the following expression for R, the curvature scalar, which we shall need later:

$$R = {}'g^{\mu\lambda}\,{}'g^{\kappa\nu} B^{\alpha\beta\varrho\delta}_{\kappa\mu\lambda\nu} R_{\alpha\beta\varrho\delta} + 2R^{\kappa\nu} n^{\kappa} n^{\nu}, \tag{4.12}$$

it is quite simple to show, with the help of the Gauss equation (4.6), that:

$$n^{\mu} n^{\nu} G_{\mu\nu} = -\frac{1}{2} B^{\alpha\beta\varrho\delta}_{\mu\nu\kappa\lambda} R_{\alpha\beta\varrho\delta}\,{}'g^{\nu\lambda}\,{}'g^{\mu\kappa}. \tag{4.13}$$

Using the Gauss equation again, we get:

$$n^{\mu} n^{\nu} G_{\mu\nu} = -\frac{1}{2} ({}'R + h_{\mu\nu} h^{\mu\nu} - h^{\mu}_{\mu} h^{\nu}_{\nu}); \tag{4.14}$$

thus the vanishing of the normal-normal component of the Einstein equations implies the constraint:

$${}'R + h_{\mu\nu} h^{\mu\nu} - h^{\mu}_{\mu} h^{\nu}_{\nu} = 0 \tag{4.15}$$

on the initial data.

Similarly, it is easily shown that

$$B^{\mu}_{\kappa} n^{\nu} G_{\mu\nu} = B^{\alpha\beta\varrho}_{\delta\lambda\kappa} R_{\beta\varrho\tau\alpha} n^{\tau}\,{}'g^{\delta\lambda}. \tag{4.16}$$

By use of the Codazzi equations (4.3), this reduces to:

$$B^{\mu}_{\kappa} n^{\nu} G_{\mu\nu} = {}'\nabla_{\alpha} h^{\alpha}_{\kappa} - {}'\nabla_{\kappa} h^{\alpha}_{\alpha}; \tag{4.17}$$

which shows that the vanishing of the normal-hyoersurface comoonents of the Einstein tensor imply the three constraints:

$${}'\nabla_{\alpha} h^{\alpha}_{\kappa} - {}'\nabla_{\kappa} h^{\alpha}_{\alpha} = 0. \tag{4.18}$$

The Lie derivatives of the constraint equations may be computed directly. They turn out to be linear combinations of the constraint equations and the remaining field equations. So, if the constraint equations hold initially, and the other field equations hold off the initial hypersurface, the constraint equations will hold off the initial hypersurface. Thus, if we pick the first fundamental form ${}'g_{\mu\nu}$ and its first Lie derivative in the unit normal direction (second fundamental form) subject to the four constraints (4.10) and (4.13); and determine the second and all higher Lie derivatives of ${}'g_{\mu\nu}$ from (4.5), we determine a solution to the empty space field equations propagating along the family of geodesically parallel hypersurfaces to our initial hypersurface.

More generally, it can be shown that the conservation law for any symmetric second rank rensor implies such a relationship between the Lie derivatives of its normal-normal and normal-hypersurface components, and the tensor itself. Thus, the propagation of the constraint equations off the initial hypersurfaces is seen to be a consequence of the conservation law $\nabla_{\nu} G^{\mu\nu}$, which is, of course, just the twice-constracted Bianchi identity.[20] Thus, if we pick the first fundamental form ${}'g_{\mu\nu}$ and its first Lie derivative in the unit normal

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direction (second fundamental form) subject to the four constraints (4.10) and (4.13); and determine the second and all higher Lie derivatives of $'g_{\mu\nu}$ from (4.5), we determine a solution to the empty space field equations propagating along the family of geodesically parallel hypersurfaces to our initial hypersurface. As mentioned above, this technique may be fairly easily generalized to use Lie derivatives with respect to an arbitrary vector field v^{μ}, neither normal to the hypersurface nor geodesic. For the details, we refer to [18].

V. Lagrangian and Hamiltonian formulations

We have seen in the previous section how one may, in a straightforward way, compute the Lie derivatives of the surface-surface components of the metric (*i.e.*, the first fundamental form of the hypersurface) and use the field equations to demonstrate that the second and all higher Lie derivatives are determined by the first and second fundamental form, when $G_{\mu\nu} = 0$. We have called this a Newtonian approach, because the Lie derivative plays much the same role as the total time derivative in Newtonian mechanics; and our procedure is similar to the use of the force law to determine second and higher derivatives of the the position, given the position and its first time derivative, the velocity. We now show that this analogy may be carried further. The variational principle for the Einstrin field equations may be reformulated in terms of Lie derivatives, which continue to behave in much the same way as total time derivatives in the Lagrangian formulation of particle mechanics. We can derive the field equations in their Lie derivative form in this way, as well as the constraint equations; and make the usual transition to a Hamiltonian formalism as well.

We start from the standard variational principle for deriving the field equation:

$$\delta \int (-g)^{1/2} R d^4x = 0, \tag{5.1}$$

where, as usual, g is the determinant of the metric tensor $g_{\mu\nu}$, and R is the four-dimensional curvature scalar. The procedure may be described somewhat more explicitly as follows: starting with some region of the bare four-dimensional manifold, we break it up into small sub-regions, symbolized by d^4x. Then we impose a metric on the whole region, compute $(-g)^{1/2}$ (which when multiplied by d^4x gives the volume of the subregions), and R in each subregion, and add (integrate) to compute the integral. The variational procedure consists in imposing various different metrics on the manifold, and seeing which class of metrics makes the integral an extremum. In particular, we may break up our region by means of a family of hypersurfaces generated from an initial hyperface by some transvecting vector field n^{μ}, then further subdividing the region between two neighboring hypersurfaces by means of the streamlines of the vector field. This gives us a definite breakup of the region. Now we may impose arbitrary hypersurface metrics on the family of surfaces and vary hypersurface metrics. Since each choice of vector field with assigned normal and surface projections, C^n_{ν} and B^{μ}_{ν}, gives rise, together with a hypersurface metric, to a full metric $g_{\mu\nu}$, we expect the variation of the hypersurface metric alone to give rise to six of the field equations-those describing the evolution of the hypersurface metric. The variation of the vector field then gives rise to the remaining four equations: the four constraint equations describing the normal-normal and hypersurface-normal projections of the field equations. If the raised

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metric is written in the form $g^{\mu\nu} = {'g}^{\mu\nu} + n^\mu n^\nu$, then this may be varied and substituted into the variation of Eq. (6.1):

$$\delta \int (-g)^{1/2} R d^4x = \int (-g)^{1/2} G_{\mu\nu} \delta g_{\mu\nu} d^4x =$$

$$\int (-g)^{1/2} [G_{\mu\nu}(\delta' g^{\mu\nu}) + 2G_{\mu\nu} n^\mu \delta n^\nu] \, d^4x = 0. \tag{5.3}$$

Variation of the hypersurface metric then leads to the equations:

$$B^{\mu\nu}_{\alpha\beta} G_{\mu\nu} = 0. \tag{5.4a}$$

Variation of the normal vector (which may be independently varied either parallel or orthogonal to its direction) leads to the remaining four equations:

$$G_{\mu\nu} B^\mu_\varkappa n^\nu = 0, \qquad G_{\mu\nu} n^\mu n^\nu = 0. \tag{5.4b}$$

Actually, all of our results up to now would hold true for any unit normal field to a family of hypersurfaces, geodesic or not. It is only when we express the variational principle in terms of Lie derivatives that the restriction to geodesic normal fields introduces great simplifications. To do this, we need to express both R and the volume element $(-g)^{1/2} d^4x$ in terms of the hypersurface metric ${'g}_{\mu\nu}$ and its Lie derivatives. From our previous results, the following identity is easily proved:

$$R = -{'g}^{\mu\nu} \underset{n}{\pounds}^2 \, {'g}_{\mu\nu} + {'R} + 3h_{\mu\nu} h^{\mu\nu} - h^\mu_\mu h^\nu_\nu. \tag{5.5}$$

For the volume elements we are using, it is obvious that $(-g)^{1/2} d^4x = (-{'g})^{1/2} d^3x \, dt$, where $\boldsymbol{dt}$ is the parameter distance between neighboring geodesically parallel hypersurfaces of the family into which we have split the region. The variational principle thus becomes:

$$\delta \int (-{'g})^{1/2} (-{'g}^{\mu\nu} \underset{n}{\pounds}^2 \, {'g}_{\mu\nu} + {'R} + 3h_{\mu\nu} h^{\mu\nu} - h^\mu_\mu h^\nu_\nu) \, d^{3\prime}x \, dt = 0. \tag{5.6}$$

If we define $L = \int \mathcal{L} \, d^{3\prime}x$, then this is of the form $\int L \, dt$; and the Lie derivatives in the integrand behave very much like total time derivatives ("velocities") in treating the variations. In particular, the field equations take the form:

$$\frac{\delta L}{\delta' g_{\mu\nu}} - \underset{n}{\pounds} \frac{\delta L}{\delta \underset{n}{\pounds}' g_{\mu\nu}} + \underset{n}{\pounds}^2 \frac{\delta L}{\delta \underset{n}{\pounds}^{2\prime} g_{\mu\nu}} = 0. \tag{5.7}$$

A total Lie derivative may be added to the integrand without altering the field equations. For our variational integrand, the second order Lie derivative term can be removed from the integrand by addition of such a total Lie derivative. Using the result:

$$\underset{n}{\pounds} (-{'g})^{1/2} = -(-{'g})^{1/2} h^\mu_\mu, \tag{5.8}$$

it is seen that the variational integrand may be written

$$\mathcal{L} = \underset{n}{\pounds} [-(-{'g})^{1/2} {'g}^{\mu\nu} \underset{n}{\pounds} \, {'g}_{\mu\nu}] + (-g)^{1/2} ({'R} - h_{\mu\nu} h^{\mu\nu} + h^\mu_\mu h^\nu_\nu). \tag{5.9}$$

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We can obviously eliminate the first of these terms by addition of a total Lie derivative, getting the new Lagrangian density

$$\overline{\mathfrak{L}} = (-g)^{1/2}('R - h_{\mu\nu}h^{\mu\nu} + h^{\mu}_{\mu}h^{\nu}_{\nu}). \tag{5.10}$$

Variation of this Lagrangian with respect to the hypersurface metric $'g_{\mu\nu}$, then results in the six equations (5.4a); while variation with respect to the normal field gives the four constraint equations (5.4b). We omit the detailed proof, which may be found in [18].

The role of the $\underset{n}{\pounds}\,'g_{\mu\nu}$ as "velocities" in the variation with respect to $'g_{\mu\nu}$ discussed above, suggests the definition of field momenta and a Hamiltonian in the usual way. We define hypersurface momentum density tensors $\tilde{p}^{\mu\nu}$ and the Hamiltonian by

$$\tilde{p} = \frac{\partial \overline{\mathfrak{L}}}{\partial' \underset{n}{\pounds} g_{\mu\nu}} = (-'g^{1/2})(h^{\mu\nu} - 'g^{\mu\nu}h^{\varkappa}_{\varkappa}) \tag{5.11}$$

and

$$\tilde{H} = \tilde{p}^{\mu\nu}\underset{n}{\pounds}\,'g_{\mu\nu} - \overline{\mathfrak{L}} = -(-'g)^{1/2}('R + h_{\mu\nu}h^{\mu\nu} - h^{\mu}_{\mu}h^{\nu}_{\nu})$$
$$= -(-'g)^{1/2}\left['R + (-'g)^{-1}\left(\tilde{p}^{\mu\nu}\tilde{p}_{\mu\nu} - \frac{1}{2}\tilde{p}^{\mu}_{\mu}\tilde{p}^{\nu}_{\nu}\right)\right]. \tag{5.12}$$

As is well known,[3] the resulting Hamiltonian equations of motion

$$\underset{n}{\pounds}\,'g_{\mu\nu} = \frac{\partial \tilde{H}}{\partial \tilde{p}^{\mu\nu}}, \qquad \underset{n}{\pounds}\tilde{p}^{\mu\nu} = -\frac{\partial \tilde{H}}{\partial' g^{\mu\nu}}, \tag{5.13}$$

are indeed equivalent to the definition of the momenta, and the equations of motion $G_{\mu\nu}B^{\mu\nu}_{\alpha\beta} = 0$, respectively. Thus, we see that $H = \int \tilde{H}\, d^2x$, which has been called the main part of the Hamiltonian by Dirac [7], governs the evolution of the hypersurface metric on a family of geodesically parallel hypersurfaces. $\tilde{H} = 0$ is itself one of the constrait equations, of course. The other set of constraint equations may be rewritten in terms of the momenta as $'\nabla_{\mu}\tilde{p}^{\mu\nu} = 0$.

VI. Constraint equations

As mentioned in Section IV, the vector field with which we rig the initial hypersurface may be chosen and continued off the initial hypersurface in an arbitrary way. The Cauchy problem for the family of hypersurfaces generated by dragging along the initial hypersurface by this vector field may then be formulated and formally solved by methods analogous to those of Section IV [18].

The use of a geodesic normal field merely serves to simplify the discussion as much as possible and highlight the geometrical content of the work. But it is worthwhile to examine the problem of the constraints on the initial data from the point of view of an arbitrary rigging field v^{μ} for the light it sheds on the meaning of these equations. The vector field v^{μ} can always be written in the form $v^{\mu} = \varrho\, n^{\mu} + \sigma^{\mu}$, where n^{μ} is the normal vector to the surface, and σ^{μ} is a vector field in the hypersurface: $v^{\mu}\sigma_{\mu} = 0$. The vector $\varrho\, n^{\mu}$ dt generates a transformation in the normal direction to an arbitrary first infinitesimally neighboring

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hypersurface; while $\sigma^\mu\, dt$ generates an arbitrary displacement within the hypersurface. Between them then, they generate an arbitrary transformation between the initial hypersurface and any first infinitesimally neighboring hypersurface. Suppose we are given the first fundamental form $'g_{\mu\nu}$ on the initial hypersurface, and its Lie derivative with respect to v^μ on that hypersurface. This is equivalent to giving the first fundamental form on both the initial hypersurface and the first infinitesimally neighboring hypersurface generated by dragging along $v^\mu\, dt$. We can express $\underset{v}{\pounds}$ in terms of $\underset{\varrho n}{\pounds}$ and $\underset{\sigma}{\pounds}$ since the Lie derivative with respect to the sum of two fields is just the sum of the Lie derivatives with respect to these fields separately. When we apply this result to $'g_{\mu\nu}$, we can easily show that

$$\underset{v}{\pounds}\, 'g_{\mu\nu} = \varrho \underset{n}{\pounds}\, 'g_{\mu\nu} + \underset{\sigma}{\pounds}\, 'g_{\mu\nu}, \tag{6.1}$$

But $\underset{\sigma}{\pounds}\, 'g_{\mu\nu}$, is just the Lie derivative of the metric tensor of the surface with respect to a vector field on the surface, and this is just the Killing form of the vector field (See the Appendix for proof):

$$\underset{\sigma}{\pounds}\, 'g_{\mu\nu} = '\nabla_\mu \sigma_\nu + '\nabla_\mu \sigma_\nu. \tag{6.2}$$

Using these results, and remembering that $\underset{v}{\pounds}\, 'g_{\mu\nu} = -2h_{\mu\nu}$, we can express the second fundamental form on the initial hypersurface as

$$h_{\mu\nu} = (1/2\, \varrho)\, (-\underset{v}{\pounds}\, 'g_{\mu\nu} + '\nabla_\mu \sigma_\nu + '\nabla_\mu \sigma_\nu). \tag{6.3}$$

We may now insert this expression for $h_{\mu\nu}$ into the constraint equations (4.10) and (4.13) so that the latter are re-expressed in terms of the Lie derivatives of $'g_{\mu\nu}$ with respect to v^μ, ϱ and σ^μ. These equations may be looked upon in two different ways. If we regard ϱ and σ^μ as given, *i. e.*, if the vector field v^μ is given, they are equations constraining the choice of $\underset{v}{\pounds}\, 'g_{\mu\nu}$. Geometrically, this means that, if we regard the location and relationship of the neighboring hypersurfaces as given, the metric of the first infinitesimally neighboring hypersurface may not be picked arbitrarily, but only subject to these constraints. But, if we regard $\underset{v}{\pounds}\, 'g_{\mu\nu}$ as given to us, the constraint equations may be looked upon as equations for ϱ and σ^μ, i. *e.*, for the vector field v^μ. Geometrically, this means that we may prescribe the metric on both infinitesimally neighboring surfaces, but look upon the constraint equations as conditions on the "fit" of the first infinitesimally neighboring hypersurface relative to the initial hypersurface: ϱ determines the surface, and σ^μ determines the relationship between points on the two surfaces. Explicitly, the equations take the form:

$$4\varrho^2\, 'R + (-\underset{v}{\pounds}\, 'g_{\mu\nu} + '\nabla_\mu \sigma_\nu + '\nabla_\nu \sigma_\mu)\, (-\underset{v}{\pounds}\, 'g_{\varkappa\lambda} + '\nabla_\varkappa \sigma_\lambda + '\nabla_\lambda \sigma_\varkappa)$$
$$\times ('g^{\mu\varkappa}\, 'g^{\mu\lambda} - 'g^{\mu\nu}\, 'g^{\varkappa\lambda}) = 0 \tag{6.4}$$

$$'g^{\beta\nu} \left\{ '\nabla_\beta \left[\frac{1}{2p} (\underset{v}{\pounds}' g_{\mu\nu} + '\nabla_\mu \sigma_\nu + '\nabla_\nu \sigma_\mu) \right] - '\nabla_\nu \left[\frac{1}{2} (-\underset{v}{\pounds}' g_{\beta\nu} + '\nabla_\beta \sigma_\nu + '\nabla_\nu \sigma_\beta) \right] \right\} = 0 \tag{6.5}$$

Wheeler[3] has studied this way of looking at the constraint equations in a coordinate system adapted to the v^μ field: $v^\mu \stackrel{*}{=} \delta^\mu_0$. In such a coordinate system $\varrho \stackrel{*}{=} (g^{00}_{00})^{-1/2}$ and $\sigma_b \stackrel{*}{=} -g_{0b}$. He has further conjectured that it may be possible to allow a *finite* separation between the two hypersurfaces, prescribe two arbitrary positive-definite metrics on them, and then fill in the intervening region of space-time so that $G_{\mu\nu}$ vanishes everywhere on and between the two hypersurfaces. However, there are reasons for doubting whether this can be done in general.[21]

VII. Discussion

The advantage of the formalism presented here lies in its covariant, geometric nature, and the systematic way in which the results are obtained, without need to resort to special methods in each case. We have not explicitly indicated the form of our equations in an adapted coordinate system; but, by noting that when $v_\mu \stackrel{*}{=} \delta^\mu_0$, $\underset{v}{£} \stackrel{*}{=} \partial_0$, they are easily worked out. In this form they can then be compared with the usual results on the Cauchy problem, both the classical early work,[2] and the recent work on Lagrangian and Hamiltonian forms.[3] See [18] for further details.

It seems clear that an extension of the ideas of this paper to a covariant formulation of recent work on the Cauchy problem on a null hypersurface [16], [2], [14] could be made; we hope to return to this problem.

The method of treating the vanishing of the divergence of a symmetric tensor by converting it to an equation for the Lie derivatives of the normal-normal and surface-normal components of the tensor with respect to some vector field can be used to solve the three hypersurface constraint equations, as mentioned in Section IV. By introducing a conformal factor into the metric of a family of two-dimensional geodesically parallel surfaces on the initial hypersurface, one can then satisfy the remaining constraint equation. In this way, it can be shown that the conformal metric on a family of geodesically parallel two-dimensional surfaces on the initial hypersurface, together with the Lie derivative of this conformal metric with respect to the geodesic normal field, can be used (together with certain additional data on one two-dimensional surface) to determine a solution to the constraint equations. This result parallels the work of Sachs [16] on the initial value problem on null hypersurfaces.

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APPENDIX

In the Appendix we shall briefly discuss some of the mathematical concepts used in this paper which may be unfamiliar to some physicists, such as dragging a geometrical object along a covariant vector field, the dragged along field, and the Lie derivative. We shall also

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prove some results used in the text. We shall follow Schouten's notation. His discussion of these concepts is recommended for a more detailed treatment [17].

Any contravariant vector field $\boldsymbol{v}$ throughout some region of a manifold may be used to generate the infinitesimal point transformation $\boldsymbol{x} \rightarrow \boldsymbol{x}' = \boldsymbol{x}+\boldsymbol{v}\, dt$, where dt is an infinitesimal scalar parameter (we suppress indices where unnecessary). By iterating the infinitesimal point transformation generated by $\boldsymbol{v}$, we get a one-parameter Lie group of point transformation $\boldsymbol{x}(0) \rightarrow \boldsymbol{x}(t)$, where $\boldsymbol{t}$ is the scalar parameter labeling the transformations.

A geometrical quantity may be defined as a set of numbers Φ_A (A stands for all indices involved) defined at a point in the manifold obeying a transformation law more general than that of a tensor: the Φ_A form geometrical quantity if their values at a point in any coordinate system may be computed from their values at that point in any other coordinate system by an expression which is homogeneous linear in the Φ_A, and homogeneous algebraic in the derivatives of the functions describing the coordinate transformation between the two coordinate systems. When we carry out any of the point transformations discussed above, we can carry along the values of any geometrical quantity field defined over the region of the manifold under consideration. This is called dragging along the field, and defines a new field in the region, of the same type as the original field, called the dragged-along field $\overset{m}{\Phi}_A(t)$. More precisely, the dragged-along field may be defined in terms of that unique coordinate transformation, such that the coordinates of the new point $\boldsymbol{x}(t)$ in the *new* coordinate system (symbolized by a prime) equal the coordinates of the old point $\boldsymbol{x}(0)$ in the old coordinate system:

$$x(t)^{\beta'} \overset{*}{=} x(0)^{\beta} \tag{A.1}$$

(the star on an equation means that it is true only in certain coordinate systems). Then the dragged-along field $\overset{m}{\Phi}_A$ has values at the *new* point in the *new* coordinate system that are numberically equal to the original field values at the old point in the old coordinate system.

$$\Phi_A{}' \,[x(t)] \overset{*}{=} \Phi_A[x(0)] \tag{A.2}$$

The one-parameter group of point transformations thus defines a one-parameter family of dragged-along field $\overset{m}{\Phi}_A(t)$, such that $\overset{m}{\Phi}_A(0) = \Phi_A$. We can now compare the values of the original field and any of the dragged-along fields at the same point (in the same coordinate system, of course). The difference of these two values will clearly go to zero as $t \rightarrow 0$, and we define the Lie derivative of the field Φ_A with respect to $\boldsymbol{v}$ as the limit of the ration of this difference divided by t as $t \rightarrow 0$:

$$\underset{v}{\pounds}\Phi_A = \lim_{t\to 0} \frac{\Phi - \overset{m}{\Phi}_A(t)}{t} \tag{A.3}$$

It is easily seen that $\underset{v}{\pounds}\, \Phi_A$ will be the same kind of geometrical quantity as Φ_A[17].

If we know the Lie derivatives of Φ_A with respect to $\boldsymbol{v}$ to all orders at some point on

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a trajectory of the vector field $\boldsymbol{v}$, we can find the dragged along fields by exponentiation of the Lie derivative:

$$\overset{m}{\Phi}_A(t) = e^{\underset{v}{t\pounds}} \Phi_A(0) = \left[1+t\underset{v}{\pounds} + \frac{t^2}{2!}\pounds^2+\dots\right] \Phi[0]. \tag{A.4}$$

Thus, when we want to compare the values of a field at different points along a trajectory of the vector field $\boldsymbol{v}$ in a covariant way, we need only drag the field the desired parameter distance along the vector field, in either direction. For the Cauchy problem, we are interested in comparing the values of certain fields on a family of hypersurfaces $\varphi =$ constant which are generated from an initial hypersurface $\varphi(0)$ by the one-parameter family of point transformations generated by $\boldsymbol{v}$. To do this covariantly, we need merely drag back the fields on the hypersurface $\varphi(t)$ to the initial hypersurface $\varphi(0)$, so that:

$$\overset{m}{\Phi} [\varphi(0)] = e^{-t\underset{v}{\pounds}} \Phi[\varphi(t)] \tag{A.5}$$

We see that a knowledge of the Lie derivatives to all orders on the initial hypersurface is what is needed to be able to drag back the fields a finite parameter distance. Thus, the Cauchy problem may be covariantly formulated in terms of the use of the field equations to determine higher order Lie derivatives, given certain initial data on the fields and some of their Lie derivatives on the initial hypersurface.

The Lie derivative has a number of properties which we shall use. We merely list them here, referring to Schouten [17] for proofs:

a. In any coordinate system in which $v^\mu \overset{*}{=} \delta^\mu_0$ (such coordinate systems always exist locally for a contravariant vector field), the Lie derivative of a geometrical quantity reduces to the ordinary derivative in the x^0 direction: $\underset{v}{\pounds}\Phi_A \overset{*}{=} \partial_0\Phi_A$. It is even possible to base the definition of the Lie derivative on this property[10].

b. The Lie derivative commutes with the ordinary partial derivative (note that it ordinarily does not commute with the operation of raising and lowering indices).

c. The Lie derivative obeys Leibnitz' rule for products: the Lie derivative of a product of two terms is the sum of each of the terms times the Lie derivative of the term. This rule may be used to derive the expression for the Lie derivatives of higher order tensors from the rules for vectors, scalars and scalar densities given below.

d. If a is a scalar field, $\boldsymbol{b}$ a scalar density field, w_μ a covariant vector field, and v^μ a contravariant vector field, their Lie derivatives are given by:

$$\underset{v}{\pounds}\, a = v^\mu a,_\mu; \quad \underset{v}{\pounds}\tilde{b} = v^\mu b,_\mu + \tilde{b}\, v^\mu,_\mu; \tag{A.6}$$

$$\underset{v}{\pounds}\, w_\mu = v^\nu w_{\mu,\nu} + w_\nu v^\nu,_\mu \underset{v}{\pounds}\, k^\mu = v^\nu k^\mu,_\nu - k^\nu v^\mu,_\nu.$$

e. In the expression for the Lie derivative of any tensor, the ordinary derivatives may be changed to covariant dericatives with respect to any affinity without changing the value of the expression.

f. The Lie derivative with respect to the vector field v of the metric of a Riemannian space is the Killing form of the vector field v:

$$\underset{v}{\pounds}\, g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu. \tag{A.7}$$

A hypersurface of a manifold may be defined by an equation of the form $\varphi(x^\lambda) = \text{const.}$ Such a hypersurface is said to be rigged by a vector field v^μ, whenever such a vector field is given over the hypersurface which transvects it at each point; *i. e.*, such that $v^\mu \varphi,_\mu \neq 0$ at any point of the hypersurface. Projection operators onto such a hypersurface and onto the rigging field may be defined by $B^\mu_\nu = \delta^\mu_\nu - v^\mu v_\nu$ and $C^\mu_\nu = v^\mu v_\nu$, respectively. Any tensor index may then be broken up into components in the hypersurface and rigging field directions by these projection operators.

The first fundamental form, or inner metric, of a hypersurface embedded in a Riemannian space, is the metric induced between points on the hypersurface by the metric of the full space. It is easily proved that $'g_{\mu\nu}$, the first fundamental form is given by

$$'g_{\mu\nu} = B^{\varkappa\lambda}_{\mu\nu} g_{\varkappa\lambda}, \tag{A.8}$$

where $g_{\varkappa\lambda}$ is the metric of the space, and $B^{\varkappa\lambda}_{\mu\nu} = B^\varkappa_\mu B^\lambda_\nu$, with $B^\varkappa_\mu$ the projection operator onto the hypersurface, given by $B^\varkappa_\mu = \delta^\varkappa_\mu - n^\varkappa n_\mu$, where $n^\varkappa$ is the unit normal to the hypersurface. $c^\mu_\nu = n^\mu n_\nu$ is the projection operator onto the normal direction.

The second fundamental form, or extrinsic metric, of a hypersurface is a measure of the bending of the hypersurface as embedded in the Riemannian space. It is defined as

$$h_{\mu\nu} = -B^{\varkappa\lambda}_{\mu\nu} \nabla_{(\varkappa} n_{\lambda)}, \tag{A.9}$$

i. e., the projection of the symmetrized covariant derivative of the normal onto the hypersurface. We shall now prove that the Lie derivative of the first fundamental form of a hypersurface with respect to the unit geodesic normal field to the hypersurface equals minus twice the second fundamental form.

First, note that $'g_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, so that

$$\underset{n}{\pounds}\, 'g_{\mu\nu} = \underset{n}{\pounds}\, g_{\mu\nu} - n_\mu \underset{n}{\pounds}\, n_\nu - n_\nu \underset{n}{\pounds}\, n_\mu. \tag{A.10}$$

Now,

$$\underset{n}{\pounds}\, n_\mu = n^\nu \nabla_\nu n_\mu + n_\nu \nabla_\mu n^\nu, \tag{A.11}$$

and for a unit geodesic normal field, both these terms vanish. On the other hand, the Lie derivative of $g_{\mu\nu}$ is the Killing form of n_μ (see rule *f.* above):

$$\underset{n}{\pounds}\, g_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu. \tag{A.12}$$

But for a geodesic normal field, the Killing form has no projections in the direction of the field itself, so that

$$\underset{n}{\pounds}\, g_{\mu\nu} = B^{\varkappa\lambda}_{\mu\nu}(\nabla_\varkappa n_\lambda + \nabla_\lambda n_\varkappa) = -2h_{\mu\nu}. \tag{A.13}$$

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NOTES

[1] Mathematical discussions of the Cauchy problem, with applications to physics, may be found in [9], [10], [4].

[2] Some important early papers on the Cauchy problem in general relativity include [5], [12], [19]; [13] summarizes earlier work of Lichnerowicz.

[3] Some recent contributions on these problems include [7]; R. Arnowitt, S. Deser, and C. Misner, *The Dynamics of General Relativity*, in [21]; P. G. Bergmann, *The General Theory of Relativity*, in [8]; J. A. Wheeler, *Geometrodynamics and the Issue of Final State*, in [6].

[4] For a more detailed discussion, see the author's Dissertation, [18].

[5] A detailed discussion of these problems for general relativity with references to earlier work is found in Y. Bruhat, *The Cauchy Problem*, in [21]. T. Taniuti [19a] gives a survey of a number of local and global problems connected with non-linear wave propagation, with applications to general relativity.

[6] These results are proved in [10], for example.

[7] We follow the notation of [17], which also contains fuller explanations of many mathematical concepts used.

[8] For a discussion of the Cauchy problem for a theory invariant under a gauge group, see [1], [9].

[9] See the Appendix or [17] for definitions of rigged hypersurface, dragging along with a point transformation, and the Lie derivative.

[10] R. K. Sachs, *Gravitational Radiation*, in [6] shows how the Lie derivative may be defined using this idea.

[11] It has been noted a number of times that Maxwell's equations may be formulated without a metric. See [15] for a detailed discussion and bibliography.

[12] By Born-Infeld type theory, we mean a theory of electrodynamics based on any Lagrangian depending on the two invariants of the electromagnetic field, rather than confining ourselves to the particular Lagrangian discussed by Born and Infeld.

[13] See [20], pp. 192-196 for bibliographical references to the early history of this work.

[14] We might equivalently use $F_{\mu\nu} = 1/2\, \varepsilon_{\mu\nu\varkappa\lambda}\tilde{F}^{\varkappa\lambda}$, and the dual of equation (3.1): $\partial_{[\mu}F_{\varkappa\lambda]} = 0$ to get the usual correspondence between $F_{\mu\nu}$ and $\vec{E}$ and $\vec{B}$. But $\tilde{F}^{\varkappa\lambda}$ is more convenient for our purposes. Note that a tilde over any quantity indicates that it is a tensor density of weight one.

[15] Further discussion of this question will be found in J. Stachel, *Comments on Casuality Requirements and the Theory of Relativity* in [3].

[16] See the Appendix or [17] for a discussion of the first and second fundamental forms of a hypersurface in a Riemannian space.

[17] See the Appendix for a proof of this relationship.

[18] We use primes in front of tensors and covariant derivatives to indicate that these are the projections of the corresponding tensors onto a hypersurface of the four-dimensional manifold. One could just as well use a separate coordinates system on the hypersurface, of course. See [18] for details.

[19] Of course, the question of the existence of a solution in the nonanalytic case requires special treatment (see note 5 for a reference). The procedure outlined above may be generalized to treat the Cauchy problem with sources present, and to treat the Cauchy problem in the tetrad formalism (see [18] for further details.)

[20] This technique may be applied to the solution of the three constraint equations (4.18), which take the form of the divergence of a three-dimensional tensor. We discuss this briefly in the concluding section.

[21] See comments by P. G. Bergmann on J. A. Wheeler, *Mach's Principle as a Boundary Condition for Einstein's Field Equations and as a Central Part of the "Plan" of General Relativity*, both in [11]. Such questions, and indeed the question of the conditions under which eqs. (6.4) and (6.5) may be solved in some region require further study of the differential equations that result when we write these equations in an adapted coordinate system.

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