ON THE S-MATRIX UNITARITY PROBLEM

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The S-matrix unitary problem is discussed in the axiomatic framework of quantum field theory. The necessary and sufficient condition, in terms of the aperture of the asymptotic subspaces, for unitatiry of the S-matrix is proposed.

1. Introduction

The well known Wightman–Gårding axioms [1] for quantum field theories are probably incomplete as it is indicated by the S-matrix unitarity problem.

It has been proves by Haag [2] and Ruelle [3] that it is possible, in a theory with non--zero minimum mass, to find in the Hilbert space $\mathcal{H}$ od states the two subspaces $\mathcal{H}_-$ and $\mathcal{H}_+$ which may be physically interpreted as being the spaces of asymptotic states describing particles moving before and after collision. Moreover, it also has been shown that there exists an isometric mapping of $\mathcal{H}_+$ onto $\mathcal{H}_-$, which can be extended to a partially isometric S operator in $\mathcal{H}$ such that

$$S^+S = P_+,\ SS^+ = P_- \tag{1.1}$$

where $P_\pm$ are projection operators for mapping on $\mathcal{H}_\pm$, respectively.

To be more specific, let us consider a one scalar neutral field $A(x)$ satisfying the usual Wightman–Gårding axioms. Then the intermediate scattering states are introduced as strong limits of the vectors

$$\Psi_t(f_1, \ldots) = \prod_k B^*(t; f_k)\Psi_0 \tag{1.2}$$

where

$$B(t; f_k) = i \int_{x^0=t} dx f_k^*(x)\, \overleftrightarrow{\partial_0} B(x) \tag{1.3}$$

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$\{f_k(x), f_k^*(x)\}$ is a complete and orthogonal set of smooth solutions of the Klein–Gordon equation. $B(x)$ is the Jost field [4] solving the one particle problem

$$B(x) = \int A(x-y)k(y)dy$$

$$k(y) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \Delta^{(1)}(y, \varkappa^2)\, h(\varkappa^2) d\varkappa^2. \tag{1.4}$$

Here, $h(\lambda)$ is a smooth real function of the $\mathcal{D}$ class localized around the point $\lambda = m^2$, and $\Delta^{(1)}(y, \varkappa^2)$ is a known invariant function. As a result, $B(x)$ has the property

$$(\Psi_0, B(x)B(y)\Psi_0) = -i\Delta_+(x-y; m^2) \tag{1.5}$$

and also

$$\mathcal{S}-\lim_{\pm t\to\infty} \Psi_t(f_1, \ldots) = \Psi_\pm(f_1, \ldots).$$

The asymptotic subspaces $\mathcal{H}_+$ have the Fock structure

$$\mathcal{H}_\pm = \bigoplus_{n=0}^{\infty} \mathcal{H}_\pm^{(n)}$$

$$\mathcal{H}_\pm^{(n)} = \overline{L}\{\Psi_\pm(f_{k_1}, \ldots, f_{k_n})\} \equiv \overline{D_\pm^{(n)}}. \tag{1.6}$$

The S-operator is defined as a linear extension of the norm preserving mapping

$$S\Psi_+(f_1, \ldots) = \Psi_-(f_1, \ldots). \tag{1.7}$$

It is essential from the physical point of view for the S-operator to be unitary. Therefore, $\mathcal{H}_+$ and $\mathcal{H}_-$ should coincide. However, until now nobody has succeded in proving this by starting from the first axioms.

The common way of making the S-operator unitary is to postulate additionally that $\mathcal{H}_-$ (or $\mathcal{H}_+$) is complete, *i.e.*, $\mathcal{H}_- = \mathcal{H}$ [3]. Then, by means of the TCP invariance

$$\mathcal{H}_+ = \theta\mathcal{H}_- = \theta\mathcal{H} = \mathcal{H} = \mathcal{H}_-. \tag{1.8}$$

Here θ is the TCP antiunitaryoperator defined by

$$\theta A(x)\theta = A(-x)$$

$$\theta\Psi_0 = \Psi_0. \tag{1.9}$$

Its action on $\Psi_t(f_1, \ldots)$ is simple and yields

$$\theta\Psi_t(f_1, \ldots) = \Psi_{-t}(f_1', \ldots) \tag{1.10}$$

where

$$f_k'(x) = f_k^*(-x).$$

Incidentally, when $f_k(x)$ are chosen in such a way that $f_k'(x) = \pm f_k(x)$ we then have

$$||\Psi_t(f_1, \ldots)|| = ||\Psi_{-t}(f_1, \ldots)||$$

and hence

$$\frac{d}{dt} ||\Psi_t(f_1, \ldots)|| \Big|_{t=0} = 0. \tag{1.11}$$

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Clearly, the weaker assumption $\mathcal{H}_+ = \mathcal{H}_-$ would suffice for the unitarity of the S-matrix, whereas the requirement $\mathcal{H}_- = \mathcal{H}$ is too strong and, as is known [5], is independent of the first axions.

Since it is rather doubtful that the coicidence of $\mathcal{H}_+$ and $\mathcal{H}_-$ follows from the first axioms, we shall suggest a sufficient condition for this coincidence. This condition, however, is formulated in terms of the aperture of the asymptotic subspaces only. Hence, further "logical filtration" is needed for deriving an additional postulate on the basic field $A(x)$.

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2. *The aperture of linear manifolds*

The notion of the aperture of two linear manifolds in $\mathcal{H}$ is adequate for describing the relations between them. Namely, if D_1 and D_2 are linear subsets in $\mathcal{H}$, then the aperture between them is

$$A(D_1, D_2) = ||P(\bar{D}_1)-P(\bar{D}_2)|| = \sup_{h\in\mathcal{H}} \frac{||[P(\bar{D}_1)-P(\bar{D}_2)]h||}{||h||} \tag{2.1}$$

$P(\bar{D}_j)$ is the projection on the closure of D_j.

The aperture has the following basic properties [6]:

1) $$A(D_1, D_2) = A(\bar{D}_1, \bar{D}_2) = A(D_2, D_1),$$

2) $$A(D_1, D_2) = A(\mathcal{H}\ominus\bar{D}_1, \mathcal{H}\ominus\bar{D}_2),$$

3) $$0 \leqslant A(D_1, D_2) \leqslant 1.$$

4) $A(D_1, D_2) = 1$ if one of the D_j, say D_1, contains an element which is orthogonal to all elements of D_2.

5) If $A(D_1, D_2) < 1$ then $\dim D_1 = \dim D_2$,

6) The following representation is true

$$A(D_1, D_2) = \max\{\varrho_1, \varrho_2\}$$

where

$$\varrho_1 = \sup_{h\in\bar{D}_2} \frac{||[1-P(\bar{D}_1)]h||}{||h||}$$

$$\varrho_2 = \sup_{h\in\bar{D}_1} \frac{||[1-P(\bar{D}_2)]h||}{||h||}.$$

In the particular case when $D_1 \subset D_2$ one has $A(D_1, D_2) = \varrho_1$.

For example, let us consider a two dimensional plane and the one dimensiomal subspaces generated by the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ with the angle φ between then, $(\boldsymbol{e}_1, \boldsymbol{e}_2) = \cos\varphi$. Then

$$D_1 = \{\lambda \boldsymbol{e}_1\},\ D_2 = \{\mu \boldsymbol{e}_2\} \quad (\lambda \text{ and } \mu \text{ real}) \tag{2.2}$$

Simple and straightforward calculations give

$$A(D_1, D_2) = \sqrt{\max_{\Psi} f(\Psi)} \tag{2.3}$$

where

$$f(\Psi) = \cos^2 \Psi + \cos^2(\Psi - \varphi) - 2 \cos \varphi \cdot \cos \Psi \cdot \cos (\Psi - \varphi). \tag{2.4}$$

The extremum point, $\Psi = \varphi$, yields for the aperture

$$A(D_1, D_2) = |\sin \varphi|. \tag{2.5}$$

Hence, the aperture has maximum when subspaces are orthogonal, and zero when they coincide.

3. *The S-matrix unitarity condition*

Let G denote the closed linear span generated by D_+ and D_- together,

$$G = \overline{L}\{D_+ \cup D_-\}. \tag{3.1}$$

This set is relativistically invariant because the sets are invariant: $U(a, \Lambda)D_\pm = D_\pm$.

It is not difficult to see that

$$A(D_+, G) = A(D_-, G) \tag{3.2}$$

Indeed, according to the 6-th property we have

$$A(D_\pm, G) = \frac{\sup}{h \in G} \frac{||[1-P(\mathcal{H}_\pm)]h||}{||h||} \tag{3.3}$$

since $D_\pm \subset G$.

On the other hands, we have the relations

$$\theta P(\mathcal{H}_\pm)\theta = P(\mathcal{H}_\pm). \tag{3.4}$$

Therefore we may write

$$A(D_+, G) = \frac{\sup}{h \in G} \frac{||\theta[1-P(\mathcal{H}_-)]\theta h||}{||h||} = A(D_-, G).$$

In the last equality we have utilized the invariance of G under the θ-operator: $\theta G = G$.

Now we shall state our unitarity condition.

Theorem. If $A(D_+, G) < 1$, the S-matrix is unitary. Conversely, if the S-matrix is unitary, then $A(D_+, G) = 0$.

Proof: Let $\mathcal{H}_+$ be different from $\mathcal{H}_-$, *i.e.*, let

$$C = \mathcal{H}_- \setminus \mathcal{H}_+ \neq \emptyset. \tag{3.5}$$

Therefore, there exist vectors in C which are linearly independent with respect to the vectors in $\mathcal{H}_+$. But $C \subset G$, as G may be represented as $G = \overline{L}(\mathcal{H}_+ \cup C)$. Hence, the set G contains vectors which are orthogonal to the whole $\mathcal{H}_+$. According to the 4-th property of the aperture we conclude that $A(D_+, G) = 1$, what contradicts the assumption made at the outset. Hence, $C = \emptyset$, $\mathcal{H}_+ = \mathcal{H}_-$ and the S-matrix is a unitary operator. The inverse is obvious, if $\mathcal{H}_+ = \mathcal{H}_-$ then $P(\mathcal{H}_+) = P(\mathcal{H}_-) = P(G)$. According to the definition (2.1), the aperture $A(D_+, G)$ vanishes identically.

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We shall note that the our unitarity condition has a relativistically invariant sense bacause G and D_+ are relativistically invariant.

Incidentally, we note that the asymptotic completeness condition may be expressed in terms of aperture as

$$A(D_+, D) < 1. \tag{3.6}$$

The same is then ture for $A(D_-, D)$, since they are equal to each other.

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