

ANALYSIS OF THE STANDARD LONG-WAVELENGTH LOW-TEMPERATURE APPROXIMATIONS IN THE SPIN-WAVE THEORY OF THE HEISENBERG FERROMAGNET. PART I. WALLACE'S PERTURBATION METHODS

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The adequacy of the standard long-wavelength low-temperature approximations in the spin-wave theory of the isotropic Heisenberg ferromagnet is analysed, by employing three different perturbation methods and comparing the results. The spin Hamiltonian is mapped on to the boson Hilbert space (ideal spin waves) according to Holstein's and Primakoff's mapping rules, and the first approximation in the power-series expansion for the square-root is used, which is equivalent to Dyson's first Born approximation in calculating the leading correction to the free energy originating from dynamical spin-wave interactions.

In Part I, Wallace's Hamiltonian and statistical perturbation methods are applied to the boson Hamiltonian, and his results for unrenormalized spin waves are shown to be inaccurate and obtainable in a much simpler way. It is also shown that, in this approximation, Wallace's, Dyson's, and other authors' thermodynamic results are recovered in the simplest way by applying to our Hamiltonian the standard statistical perturbation method. In Part II, the correct first-order corrections to the single-particle energies and to the system's energy levels are derived, and the thermodynamic aspect of the magnon Hilbert space is studied. Part III is devoted to the refinement of the standard low-temperature thermodynamics of the Heisenberg ferromagnet.

1. Introduction

The notion of spin waves introduced by Bloch [1] in studying the thermodynamic behaviour of the Heisenberg ferromagnet with (atomic) spin one-half, and generalized by Moeller [2] to the case of arbitrary spin, became over the years a powerful tool in treating thermodynamic phenomena of ferromagnets (as well as antiferromagnets) at low tempera-

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tures (see, *e.g.*, [3–8]). Since the pioneering work by Holstein and Primakoff [9], who first succeeded in expressing Heisenberg's spin Hamiltonian in terms of the far more convenient boson operators, Bloch's rather intuitive concept of spin waves has gradually been worked out into a well-grounded and efficient theory. The subsequent development of the spin-wave theory can perhaps best be traced on the series of papers [10–29], of which Dyson's work [12] plays in many respects a fundamental role¹.

The most important result obtained in [12] resides apparently in showing that, for an isotropic Heisenberg spin Hamiltonian with a homogeneous magnetic field included, the use of boson operators (ideal spin waves) instead of spin operators (physical spin waves) is for long wavelengths and low temperatures fully justified, as the contribution of the so-called kinematical interactions between the spin waves to any thermodynamic quantity expressed in ascending powers of the (reduced) temperature are by far negligible in comparison with those due to dynamical interactions. Yet the contribution of the latter is over a wide temperature range not only much smaller (*e. g.*, for the spontaneous magnetization less than 4% for spin one-half and less than 2% for spin 1, up to one-half the Curie temperature; see [12]) than the leading term of Bloch's linear approximation (non-interacting spin waves), but does not exceed the first and second-order corrections to Bloch's term also obtained by Dyson. This led Dyson to concluding that, in the long-wavelength low-temperature approximation to which his results are restricted, the influence of the dynamical interactions is simply negligible. On the other hand, Dyson's conclusion implies that, if these interactions are to play any significant role at all (which they eventually should — some place or other) it cannot be but for short wavelengths and/or at higher temperatures. Some aspects of this problem are dealt with in Parts II and III of the present paper.

The mapping of the spin Hamiltonian on to the space of harmonic oscillators used in [12] differs from that employed in [9]. Dyson's mapping is quite simple, seems more natural from the physical viewpoint (see [29]), and ensures the mapped Hamiltonian to be of finite and low order in the boson operators (*e.g.*, of fourth order in [12]). However, these advantages hardly facilitate the perturbation calculations (*cp.*, *e.g.*, [14] and [23]) and, besides, Dyson's mapping destroys the hermiticity of the Hamiltonian². Therefore, considerable pain has been taken to recover Dyson's results by simpler methods which would be better suited to attack more complicated Hamiltonians in the framework of the spin-wave formalism. The papers [16], [19–21], [23], [24], [27], [28], [30] might well serve as a representative sample of what has been tried and achieved in this direction.

Recently, two new perturbation methods have been worked out by Wallace [31], [32] which are particularly suited for treating many-particle systems in the occupation-number

¹ An equally thorough analysis of the same spin Hamiltonian was later carried through by Wortis [24], his standard of mathematical rigour evidently competing with Dyson's — sometimes successfully. Wortis' study certainly offers some deeper insights into the spin kinematics; however, putting all the guesswork aside his results reduce practically to those obtained by Dyson, yet the formalism used in [24] is a good deal more complicated than in [12], so much so that the three-dimensional considerations in [24] had to be confined to the case of the simple cubic lattice.

² As shown in [25], the mappings used in [9], [12] are just particular cases out of a wide class of admissible mappings (see also [29]).

representation. The statistical perturbation method developed in [32] was applied in [33] to the isotropic Heisenberg ferromagnet and shown to lead in a relatively simple way to Dyson's results, within the standard long-wavelength low-temperature approximations. Moreover, the same method (with a different perturbation parameter) was shown in (32) to work equally well at higher temperatures when applied to the same spin Hamiltonian, in which case, *e.g.*, the results obtained in [34], [35] (see also [5]) by the Green's-function technique were easily recovered. This shows the remarkable efficiency of Wallace's perturbation methods in application to the Heisenberg spin Hamiltonian. Therefore, these methods deserve further exploration as to their consistency and applicability to more complicated spin Hamiltonians.

From the latter point of view, of greatest interest would apparently be further simplifications of Wallace's methods which would pay off in preserving their efficiency when considering anisotropic spin coupling, antiferromagnets, *etc.* We shall show that, without having to restrict the calculations to the nearest-neighbour interactions a considerable simplification can already be achieved when using boson operators (ideal spin waves) instead of the far less convenient Fourier transforms of the spin operators (physical spin waves) used in [32], [33], without any loss of accuracy in the final results, and that the same results can be obtained in the simplest way by using the standard perturbation calculus. Furthermore, the inconsistencies in Wallace's Hamiltonian perturbation method [31] already pointed out in [36] are here plainly demonstrated by the fact that the first-order corrections to the system's zeroth-order energy levels calculated by Wallace's methods differ from each other, and that both are inaccurate.

The application of Wallace's methods requires the Hamiltonian to be hermitian. To meet this condition, Holstein's and Primakoff's mapping [9] is used and, for simplicity, only the first two terms in the power-series expansion for the square-root in their mapping rules are retained (see [14]). This restriction is strictly equivalent to Dyson's first Born approximation in calculating the leading dynamical correction to the free energy (see also [16]). Within these limitations, Wallace's and other authors' results are easily recovered and corrected in Part II of this paper, and the thermodynamic description of the Heisenberg ferromagnet is refined in Part III.

2. The mapping of the spin Hamiltonian

We start with the spin Hamiltonian

$$H = \sum_{\mathbf{r}\mathbf{r}'} A_{\mathbf{r}\mathbf{r}'} \sum_{i=1}^3 S_r^i S_{r'}^i + B \sum_{\mathbf{r}} S_r^3,$$

$$A_{\mathbf{r}\mathbf{r}'} = A_{\mathbf{r}'\mathbf{r}} = A(|\mathbf{r} - \mathbf{r}'|) [1 - \delta_{\mathbf{r}\mathbf{r}'}] \leq 0, \quad (A_{\mathbf{r}\mathbf{r}} = 0)$$

$$B = \mu \mathcal{H}, \tag{1}$$

where the suffix $r \equiv \mathbf{r}$ is a lattice *vector*, S_r^i denotes the i -th spin component at lattice site \mathbf{r} , $A_{\mathbf{r}\mathbf{r}'}$ is the (negative) Heisenberg exchange integral between atoms at lattice sites \mathbf{r} and \mathbf{r}' , μ is the magnetic moment per lattice atom, and \mathcal{H} denotes the homogeneous external magnetic field assumed to be parallel to the coordinate axis x_3 . (Note that in our simplified

notation double suffixes such as $rr' \equiv \mathbf{r}, \mathbf{r}'$ represent *pairs of lattice vectors* — not scalar products.) Spin operators corresponding to different lattice sites commute. In terms of the operators

$$S_r^\pm = S_r^1 \pm iS_r^2 \quad (2)$$

the Hamiltonian (1) reads

$$H = \sum_{rr'} A_{rr'} (S_r^+ S_{r'}^- + S_r^3 S_{r'}^3) + B \sum_r S_r^3. \quad (3)$$

Assuming the maximum spin eigenvalues S to be the same for all lattice atoms, we use the Holstein-Primakoff mapping [9], [14], [29] in the following approximation:

$$\begin{aligned} S_r^+ &= (S_r^-)^+ \rightarrow \sqrt{2S} a_r^+ \sqrt{1 - a_r^+ a_r / 2S} \\ &= \sqrt{2S} a_r^+ \{1 - a_r^+ a_r / 4S - \dots\}, \\ S_r^3 &\rightarrow a_r^+ a_r - S, \\ [a_r, a_{r'}^+] &= \delta_{rr'}, \quad [a_r, a_r] = 0. \end{aligned} \quad (4)$$

To account for the fact that the Hilbert space of the spin operators is finite, the infinite boson Hilbert is to be restricted to a proper subspace, *i. e.*, the occupation numbers n_r are to be confined to the interval

$$0 \leq n_r \leq 2S. \quad (5)$$

This ensures the convergence of the power series in (4) and justifies the use of successive approximations. In the present paper, we shall restrict ourselves to taking into account the first two terms of the power series expansion and, consequently, to terms in the mapped Hamiltonian up to fourth order in the boson operators. We thus have

$$\begin{aligned} H &= E_0 + H_0 + H_1, \quad E_0 = S^2 \sum_{rr'} A_{rr'} - BRS, \\ H_0 &= \sum_{rr'} \{ (B - 2S \sum_{r''} A_{rr''}) \delta_{rr'} + 2SA_{rr'} \} a_r^+ a_{r'} \equiv \sum_{rr'} C_{rr'} a_r^+ a_{r'}, \\ H_1 &= \sum_{rr'} A_{rr'} \left\{ a_r^+ a_r a_{r'}^+ a_{r'} - \frac{1}{2} a_r^+ a_r^+ a_{r'} a_{r'} - \frac{1}{2} a_r^+ a_r^+ a_r a_{r'} \right\}, \\ C_{rr'} &= C_{r'r} \quad (C_{rr} \neq 0) \end{aligned} \quad (6)$$

where R is the number of lattice atoms. It is readily seen that H is self-adjoint.

3. The ideal-spin-wave representation

In diagonalizing H_0 it is instructive to use the general Bogolyubov transformation [37], [5], [29]

$$\begin{aligned} a_r &= \sum_k (u_{rk} b_k + v_{rk}^* b_k^+), \quad b_k = \sum_r (u_{rk}^* a_r + v_{rk} a_r^+), \\ [b_k, b_{k'}^+] &= \delta_{kk'}, \quad [b_k, b_{k'}] = 0 \end{aligned} \quad (7)$$

which implies the conditions

$$\begin{aligned} \sum_{\mathbf{r}} (u_{\mathbf{r}k} u_{\mathbf{r}k}^* - v_{\mathbf{r}k} v_{\mathbf{r}k}^*) &= \delta_{kk'}, \\ \sum_{\mathbf{k}} (u_{\mathbf{r}k} u_{\mathbf{r}'k}^* - v_{\mathbf{r}k} v_{\mathbf{r}'k}^*) &= \delta_{\mathbf{r}\mathbf{r}'}, \\ \sum_{\mathbf{r}} (u_{\mathbf{r}k} v_{\mathbf{r}k} - u_{\mathbf{r}k'} v_{\mathbf{r}k}) &= 0, \\ \sum_{\mathbf{k}} (u_{\mathbf{r}k} v_{\mathbf{r}'k}^* - u_{\mathbf{r}'k} v_{\mathbf{r}k}^*) &= 0 \end{aligned} \quad (8)$$

and — due to the simplicity of H_0 — the simple set of $2R^2$ homogeneous equations

$$\sum_{\mathbf{r}} (C_{\mathbf{r}\mathbf{r}'} - E_k \delta_{\mathbf{r}\mathbf{r}'}) u_{\mathbf{r}k} = 0, \quad \sum_{\mathbf{r}} (C_{\mathbf{r}\mathbf{r}'} + E_k \delta_{\mathbf{r}\mathbf{r}'}) v_{\mathbf{r}k} = 0 \quad (9)$$

for the $2R^2$ complex coefficients $u_{\mathbf{r}k}$, $v_{\mathbf{r}k}$ and the $2R$ free-particle energies E_k . (Note that the suffix $k \equiv \mathbf{k}$ is a *vector*, and that double suffixes like $rk = \mathbf{r}, \mathbf{k}$, $kk' \equiv \mathbf{k}, \mathbf{k}'$ etc. represent *pairs of vectors* — not scalar products.) As shown in [5], equations of this type are easily solved if $C_{\mathbf{r}\mathbf{r}'}$ is a function of the difference $\mathbf{r} - \mathbf{r}'$. Strictly speaking, translational invariance is needed and hence the crystal must either be infinite or else periodic boundary conditions must be imposed upon a finite crystal and surface effects neglected. This we shall henceforth assume, which means that for any two lattice vectors \mathbf{r}' and \mathbf{r}'' and any function $f(\mathbf{r} - \mathbf{r}')$ we have

$$\begin{aligned} \sum_{\mathbf{r}} A_{\mathbf{r}\mathbf{r}'} &= \sum_{\mathbf{r}} A_{\mathbf{r}\mathbf{r}''} \equiv A_0, \\ \sum_{\mathbf{r}} A_{\mathbf{r}\mathbf{r}'} f(\mathbf{r} - \mathbf{r}') &= \sum_{\mathbf{r}} A_{\mathbf{r}\mathbf{r}''} f(\mathbf{r} - \mathbf{r}''). \end{aligned} \quad (10)$$

In that case we may try the solution

$$u_{\mathbf{r}k} = u_k \exp(i\mathbf{k}\mathbf{r}), \quad v_{\mathbf{r}k} = v_k \exp(-i\mathbf{k}\mathbf{r}) \quad (11)$$

which by virtue of (9) leads to the energies

$$E_k = \pm \sum_{\mathbf{r}} C_{\mathbf{r}\mathbf{r}'} \exp[\pm i(\mathbf{r} - \mathbf{r}')\mathbf{k}] \quad (12)$$

that are independent of \mathbf{r}' , due to (10) and (6). Depending on whether the plus or minus signs are chosen in (12) one obtains from (9) and (11) either $u_k \neq 0$ and $v_k = 0$ or $u_k = 0$ and $v_k \neq 0$, respectively. However, it follows from (8) that

$$\sum_{\mathbf{r}} |u_{\mathbf{r}k}|^2 > \sum_{\mathbf{r}} |v_{\mathbf{r}k}|^2 \quad (13)$$

which implies

$$|u_k|^2 > |v_k|^2 \quad (14)$$

Thus, the only solution is

$$u_k \neq 0, \quad v_k = 0, \quad E_k = \sum_{\mathbf{r}} C_{\mathbf{r}\mathbf{r}'} \exp[i(\mathbf{r} - \mathbf{r}')\mathbf{k}] \equiv E_k^0. \quad (15)$$

For $v_k = 0$ we have from (8) and (11)

$$\sum_{\mathbf{r}} |u_{\mathbf{r}\mathbf{k}}|^2 = \sum_{\mathbf{r}} |u_{\mathbf{k}}|^2 = R |u_{\mathbf{k}}|^2 = 1,$$

$$u_{\mathbf{k}} = \exp(i\varphi_{\mathbf{k}})/\sqrt{R}, \quad \varphi_{\mathbf{k}} = \varphi_{\mathbf{k}}^* \quad (16)$$

If the Hamiltonian is to remain self-adjoint upon the transformation (7), (11), (16), it follows that $\varphi_{\mathbf{k}} = \text{const} \equiv \varphi$. Putting $\varphi = 0$ we eventually obtain the transformation

$$a_{\mathbf{r}} = \frac{1}{\sqrt{R}} \sum_{\mathbf{k}} b_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}), \quad b_{\mathbf{k}} = \frac{1}{\sqrt{R}} \sum_{\mathbf{r}} a_{\mathbf{r}} \exp(-i\mathbf{k}\mathbf{r}). \quad (17)$$

If the remaining conditions (8) are to be satisfied (*i.e.*, if the transformations $a_{\mathbf{r}} \rightarrow b_{\mathbf{k}}$ and $b_{\mathbf{k}} \rightarrow a_{\mathbf{r}}$ in (17) are to follow from each other), we must have

$$\sum_{\mathbf{r}} \exp[i(\mathbf{k}-\mathbf{k}')\mathbf{r}] = R \delta_{\mathbf{k}\mathbf{k}'}, \quad \sum_{\mathbf{k}} \exp[i(\mathbf{r}-\mathbf{r}')\mathbf{k}] = R \delta_{\mathbf{r}\mathbf{r}'} \quad (18)$$

which means that \mathbf{k} are to be wave vectors in the reciprocal lattice and (17) turns out to be a Fourier transformation.

The transformed Hamiltonian is now easily obtained as

$$H = E_0 + H_0 + H_1, \quad H_0 = \sum_{\mathbf{k}} E_{\mathbf{k}}^0 b_{\mathbf{k}}^+ b_{\mathbf{k}},$$

$$H_1 = R^{-1} \sum_{\substack{\mathbf{k}\mathbf{k}' \\ \mathbf{k}''\mathbf{k}'''}} F_{\mathbf{k}\mathbf{k}'''} \delta_{\mathbf{k}+\mathbf{k}', \mathbf{k}''+\mathbf{k}'''} b_{\mathbf{k}}^+ b_{\mathbf{k}'}^+ b_{\mathbf{k}''} b_{\mathbf{k}'''},$$

$$2F_{\mathbf{k}\mathbf{k}'''} = 2A_{\mathbf{k}-\mathbf{k}'''} - A_{\mathbf{k}} - A_{\mathbf{k}'''} \quad (19)$$

where $\mathbf{k}+\mathbf{k}' \equiv \mathbf{k}+\mathbf{k}'$, $\mathbf{k}-\mathbf{k}''' \equiv \mathbf{k}-\mathbf{k}''' \text{ etc.}$, E_0 and $E_{\mathbf{k}}^0$ are as in (6) and (15), respectively, and

$$A_{\mathbf{k}} = \sum_{\mathbf{r}} A_{\mathbf{r}\mathbf{r}'} \exp[i(\mathbf{r}-\mathbf{r}')\mathbf{k}] = \sum_{\mathbf{r}} A_{\mathbf{r}\mathbf{r}'} \cos[(\mathbf{r}-\mathbf{r}')\mathbf{k}] = A_{\mathbf{k}}^* = A_{-\mathbf{k}} \quad (20)$$

Since (10) is assumed, $A_{\mathbf{k}}$ is independent of \mathbf{r}' .

In deriving (19) we made use of the equalities

$$\sum_{\mathbf{k}} A_{\mathbf{k}} = \sum_{\mathbf{k}} A_{\mathbf{k}+\mathbf{k}'} = 0,$$

$$\sum_{\mathbf{r}\mathbf{r}'} A_{\mathbf{r}\mathbf{r}'} \exp[i(\mathbf{r}\mathbf{k}-\mathbf{r}'\mathbf{k}')] \cong R A_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'} \quad (21)$$

that follow from (1), (10), (18) and (20). The latter equality is approximate in the sense that (10) is assumed. Due to (1), (6), (10), (15), and (20) the groundstate energy E_0 and the zeroth-order spin wave energies $E_{\mathbf{k}}^0$ can be written in the form

$$E_0 = RS(A_0 S - B), \quad E_{\mathbf{k}}^0 = B + 2S(A_{\mathbf{k}} - A_0). \quad (22)$$

Since $A_0 < 0$ and $A_0 \leq A_{\mathbf{k}} \leq -A_0$, we have

$$B \leq E_{\mathbf{k}}^0 \leq B - 4SA_0. \quad (23)$$

Thus, in order that $E_k^0 \geq 0$ for all k (free-particle energies), the external field \mathcal{H} in (1) is to be chosen non-negative³, *i. e.*, $\mathcal{H} \geq 0$. In that case, also, $E_0 < 0$.

Since the transformation (7), (17) turned eventually out to be very simple and — apart from being well known [5] — could quite easily be anticipated, it may seem there is not much sense in using the general Bogolyubov transformation as the starting point. Yet it is instructive to proceed just in this way. In the first place, the transformation has actually been *shown* to be a Fourier transformation and, besides, it is clearly seen that in finite crystals the approximation (10) is made implicitly when using from the outset the transformation (17). Secondly, the negative energies in (12) are seen to be automatically excluded — without invoking any physical arguments. Furthermore, in more complicated cases (antiferromagnets, ferrimagnets, anisotropic spin coupling) the bilinear part H_0 of the Hamiltonian contains terms of the type $a_r a_r$ and $a_r^+ a_r^+$, in which case H_0 is usually being diagonalized in two steps, by first applying a Fourier transformation and then a quasi-local linear transformation in k -space (see, *e.g.*, [3], [6], [9]). In contrast, the Bogolyubov transformation permits to do it in one step, and the necessary and sufficient condition that H_0 can be diagonalized is that it be self-adjoint [37], [5].

4. The Hamiltonian perturbation method [31]

Before applying Wallace's Hamiltonian perturbation method to the Hamiltonian (19), a remark is due on the perturbation criterion used in [31], Eq. (4.15). Should we apply the same criterion to our Hamiltonian, we could arrive at the grossly erroneous conclusion that the perturbation term H_1 is *e.g.*, small. Indeed, $F_{kk''}$ in H_1 in Eq. (19) can be rearranged into $F_{kk'k''} = A_{k-k''} - (A_k + A_{k''})/2$ and

$$\sum_{\substack{k'k'' \\ k''}} F_{kk'k''} \delta_{k+k', k''+k''} = \sum_{k'k''} F_{kk'k''} = 0 \quad (24)$$

due to (21). Of course, nothing could be more wrong. The term H_1 represents the leading part of the so-called dynamical interactions and, depending on the state, its contribution to the system's zeroth-order energy levels may even be as large as that of H_0 . This is demonstrated in Parts II and III of this paper. Here we confine ourselves to drawing attention to the fact that perturbation criteria like those used in [31] can be misleading.

a) Energy corrections

According to Eq. (2.26) in [31], the first-order corrections E_k^1 to the single-particle energies E_k^0 are obtained as follows:

$$\begin{aligned} E_k^1 &= \langle \dots n_k + 1 \dots || [H_1, b_k^+] || \dots n_k \dots \rangle / \sqrt{n_k + 1} \\ &= 2R^{-1} \{ (A_0 - A_k) \sum_{k'} n_{k'} (1 - \delta_{kk'}) + \sum_{k'} (A_{k-k'} - A_{k'}) n_{k'} \} \\ &= 2R^{-1} \{ (A_k - A_0) n_k + \sum_{k'} D_{kk'} n_{k'} \}, \\ D_{kk'} &= A_0 + A_{k-k'} - A_k - A_{k'} \end{aligned} \quad (25)$$

³ Note that our mapping (4) differs slightly from that used in [9], due to which the groundstate corresponds in our case to the spin eigenvalues $-S$. Hence the implication $\mathcal{H} \geq 0$.

where $n_{\mathbf{k}}$ is the number of spin waves with wave vector \mathbf{k} :

$$\begin{aligned} \hat{n}_{\mathbf{k}} | \dots n_{\mathbf{k}} \dots \rangle &= n_{\mathbf{k}} | \dots n_{\mathbf{k}} \dots \rangle, \quad \hat{n}_{\mathbf{k}} \equiv b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \\ | \dots n_{\mathbf{k}} \dots \rangle &= \left\{ \prod_{\mathbf{k}} (n_{\mathbf{k}}!)^{-1/2} (b_{\mathbf{k}}^{\dagger})^{n_{\mathbf{k}}} \right\} | 0 \rangle, \end{aligned} \quad (26)$$

$| 0 \rangle$ being the groundstate defined by $b_{\mathbf{k}} | 0 \rangle = 0$. The states (26) are orthonormal.

Inasmuch as the single-particle energy corrections $E_{\mathbf{k}}^1$ depend on the occupation numbers $n_{\mathbf{k}}$, Wallace's renormalization procedure cannot be applied⁴. It is however claimed in [31] that — notwithstanding the dependence of the single-particle energy corrections on $n_{\mathbf{k}}$ — one can calculate the first-order corrections to the system's energy levels (see the examples considered in Sec. IV C and Sec. V B in [31]), according to the general rule for summing corrections that involve products of two or more occupation numbers:

$$\sum_{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_G} \frac{f(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_G)}{[G - g(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_G)]!} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \dots n_{\mathbf{k}_G} \quad (27)$$

where $f(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_G)$ does not depend on the occupation numbers, and $g(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_G) =$ integer $\equiv g$ whenever g of the vectors $\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_G$ coincide ($g = 0, 2, 3, \dots, G$). Hence, the first-order energy correction E^1 following from (25) is

$$\begin{aligned} E^1 &= \sum_{\mathbf{k}} \tilde{E}_{\mathbf{k}}^1 n_{\mathbf{k}} = 2R^{-1} \left\{ \sum_{\mathbf{k}} (A_{\mathbf{k}} - A_0) n_{\mathbf{k}}^2 + \sum_{\mathbf{k}\mathbf{k}'} \tilde{D}_{\mathbf{k}\mathbf{k}'} n_{\mathbf{k}} n_{\mathbf{k}'} \right\} \\ &= R^{-1} \sum_{\mathbf{k}\mathbf{k}'} D_{\mathbf{k}\mathbf{k}'} n_{\mathbf{k}} n_{\mathbf{k}'} \end{aligned} \quad (28)$$

where the tilde over sums indicates that the rule (27) is to be applied (*i. e.*, $D_{\mathbf{k}\mathbf{k}'}$ is to be replaced by $D_{\mathbf{k}\mathbf{k}'}/[2 - g(\mathbf{k}, \mathbf{k}')]!$). Since $\langle 0 | H_1 | 0 \rangle = 0$, there is no first-order correction to the groundstate energy E_0 . By combining Eqs (22) and (28) we obtain the energy levels $E = E(\dots n_{\mathbf{k}} \dots)$ of the Hamiltonian (19) in the first approximation:

$$\begin{aligned} E &= E_0 + E^0 + E^1 \equiv E_0 + E^{01} \equiv E_{01}, \quad E_0 = RS(A_0 S - B) < 0, \\ E^0 &= \sum_{\mathbf{k}} E_{\mathbf{k}}^0 n_{\mathbf{k}} > 0, \quad E^1 = R^{-1} \sum_{\mathbf{k}\mathbf{k}'} D_{\mathbf{k}\mathbf{k}'} n_{\mathbf{k}} n_{\mathbf{k}'}, \\ E_{\mathbf{k}}^0 &= B + 2S(A_{\mathbf{k}} - A_0) > 0, \quad D_{\mathbf{k}\mathbf{k}'} = A_0 + A_{\mathbf{k}-\mathbf{k}'} - A_{\mathbf{k}} - A_{\mathbf{k}'} \end{aligned} \quad (29)$$

where $A_{\mathbf{k}}$ and A_0 are defined by Eq. (20).

When simplified to the nearest-neighbour approximation, the result (29) is exactly the same as Wallace's for unrenormalized spin waves (*cp.* Eq. (3.31) in [33]), which proves that the use of the far less convenient spin operators is pointless when neglecting kinematical interactions.

As already pointed out in [33], the first-order energy correction E^1 in (29) is the same as obtained in [21]. On the other hand, our single-particle energy corrections $E_{\mathbf{k}}^1$ differ from those obtained in [33] and — by the RPA method — in [30], as in those papers the term

⁴ Unless using the method proposed in [33] which, however, is inconsistent with the conclusions arrived at in [31] (see text following Eq. (2.27) therein).

$(A_{\mathbf{k}} - A_0) n_{\mathbf{k}}$ in Eq. (25) is missing. Yet this term should be present if (28) is to be obtained by rigorously applying the rule (27); otherwise one either arrives at a different result for E^1 , or else the step-function g in the denominator must be neglected (see however Part II for the correct forms of $E_{\mathbf{k}}^1$ and E^1). The latter is the approximation usually made by Wallace (see, e.g., Sec. IV C in [31] and Sec. 7 in [33]), which is neither necessary nor properly justified by assuming it to be of order R^{-1} and thus negligible⁵.

b) Free energy and statistical averages

The last step in the calculations is to compute — correct to first order — the free energy F and the statistical averages of relevant thermodynamic variables. When applying Wallace's simple prescription for calculating the free energy correct to first order we obtain (see Sec. II D in [31])

$$\begin{aligned}
 F &= E_0 + F^0 + F^1 \equiv E_0 + F^{01} \equiv F_{01}, \\
 F^0 &= KT \sum_{\mathbf{k}} \ln [1 - \exp(-E_{\mathbf{k}}^0/KT)] = -KT \sum_{\mathbf{k}} \ln (1 + \bar{n}_{\mathbf{k}}^0), \\
 F^1 &\cong \bar{E}^1 = R^{-1} \sum_{\mathbf{k}\mathbf{k}'} D_{\mathbf{k}\mathbf{k}'} \bar{n}_{\mathbf{k}}^0 \bar{n}_{\mathbf{k}'}^0, \\
 \bar{n}_{\mathbf{k}}^0 &= [\exp(E_{\mathbf{k}}^0/KT) - 1]^{-1}
 \end{aligned} \tag{30}$$

where K — Boltzmann's constant, T — absolute temperature, and E_0 , $E_{\mathbf{k}}^0$, $D_{\mathbf{k}\mathbf{k}'}$ are given in Eq. (29).

When specified for the nearest-neighbour approximation, this result coincides with that obtained for unrenormalized spin waves by Wallace (see Eq. (3.27) in [33]). As shown in [33], in the standard long-wavelength low-temperature approximation this result corresponds for cubic lattices precisely to Dyson's first Born approximation and recovers the results obtained by different methods in [16], [19], [20], [28].

However, three remarks are due on the approximations made in deriving formula (30), as they were not paid attention to in [33]. Firstly, it is exactly at this point where the kinematical interactions re-enter the problem in a different form, as the traces in calculating the free energy are being extended over the whole boson Hilbert space — disregarding completely the restriction (5) which implies for the occupation numbers $n_{\mathbf{k}}$ the corresponding restriction (see [29])

$$0 \leq \sum_{\mathbf{k}} n_{\mathbf{k}} \equiv N \leq 2RS. \tag{31}$$

Fortunately, this inaccuracy was shown in [12] to be negligibly small for the Hamiltonian (1).

⁵ It is rather obvious that for bosons Wallace's argument applies merely to the states in which the particles are quite uniformly distributed over the wave vectors \mathbf{k} . The larger the deviation from the uniform distribution and the smaller the number of particles $N = \sum_{\mathbf{k}} n_{\mathbf{k}}$ compared with R , the worse the approximation. The latter is precisely the case for quasi-particles like phonons and spin waves at low temperatures.

Secondly, according to [36] the formula for F^1 in (30) is based on the approximation

$$F^1 \cong -KT \ln(1 - \bar{E}^1/KT) \cong \bar{E}^1 \quad (32)$$

which is valid if

$$|\bar{E}^1| \ll KT. \quad (33)$$

This restriction shall be closer examined in Part III of this paper.

Finally, the zeroth-order statistical average \bar{E}^1 as given in (30) is obtained from (29) by simply replacing the occupation numbers n_k in E^1 by their zeroth-order statistical averages \bar{n}_k^0 as defined in (30). As pointed out in [36], the correct rule for taking zeroth-order averages of products of two occupation numbers is

$$\overline{n_k n_{k'}} = \bar{n}_k \bar{n}_{k'} (1 \pm \delta_{kk'}) + \bar{n}_k \delta_{kk'} \quad (34)$$

where the plus and minus sign is for bosons and fermions, respectively. When applying the above rule to E^1 in (29), one arrives at a result that obviously differs from \bar{E}^1 in (30), namely

$$\bar{E}^{1'} = \bar{E}^1 + R^{-1} \sum_k D_{kk} \bar{n}_k^0 (\bar{n}_k^0 + 1) \quad (35)$$

Depending on the temperature and, generally, on the energy spectrum E_k^0 and the interaction tensor $D_{kk'}$, the sum in (35) need certainly not be negligible. Strangely enough, it turns out (see Sec. 6 and Part II) that the form of F^1 in (30) is, after all, correct, in spite of the fact that neither the single-particle corrections (25) nor the corrections (28) to the system's energy levels are correct (see Part II). This is due to Wallace's simplistic rule for taking statistical averages which in this case cancels precisely the errors made in calculating E_k^1 and E^1 . There is however no guarantee that such will be the case for any Hamiltonian. This emphasizes the inconsistency in Wallace's Hamiltonian perturbation method discussed in [36] which resides in the fact that corrections like E_k^1 that depend on n_k are not true single-particle energy corrections, as the particles cannot be renormalized. This is explicitly shown in Part II where the correct expressions for E^1 and E_k^1 are derived and shown to lead to F^1 from Eq. (30) when the strict rule (34) is applied.

The statistical averages \bar{n} and \bar{m} (correct to first order) of the number of spin waves per lattice atom

$$\bar{n} = R^{-1} \sum_k \ll \hat{n}_k \gg \equiv R^{-1} \sum_k \bar{n}_k \quad (36a)$$

and of the magnetization per lattice atom

$$\bar{m} = -\mu R^{-1} \sum_r \ll S_r^3 \gg = \mu \{ S - R^{-1} \sum_k \ll \hat{n}_k \gg \} = \mu \{ S - \bar{n} \} \quad (36b)$$

where

$$\ll \bar{Q} \gg \equiv \bar{Q} = Tr [Q \exp(-H/KT)] / Tr \exp(-H/KT) \quad (37)$$

can either be calculated from the free energy (30), as

$$\begin{aligned}\hat{n}_k &\equiv b_k^+ b_k = \partial H / \partial E_k^0 = \partial H_0 / \partial E_k^0, \\ R^{-1} \sum_k \hat{n}_k &\equiv \hat{n}, \quad R^{-1} \sum_k n_k = N/R \equiv n, \\ \ll \hat{n}_k \gg &\equiv \bar{n}_k = \partial F / \partial E_k^0 = \partial F^0 / \partial E_k^0 + \partial F^1 / \partial E_k^0 \equiv \bar{n}_k^0 + \bar{n}_k^1 \equiv \bar{n}_k^{01}, \\ \ll \hat{n} \gg &\equiv \bar{n} = R^{-1} \sum_k \bar{n}_k^0 + R^{-1} \sum_k \bar{n}_k^1 \equiv \bar{n}^0 + \bar{n}^1 \equiv \bar{n}^{01}\end{aligned}\quad (38)$$

and

$$\begin{aligned}\hat{M} &= -\mu \sum_r S_r^z = \mu \{RS - \sum_k \hat{n}_k\} = -\partial H / \partial \mathcal{H} = -\partial E_0 / \partial \mathcal{H} - \partial H_0 / \partial \mathcal{H}, \\ \hat{m} &\equiv \hat{M} / R \equiv R^{-1} \sum_k \hat{m}_k, \quad \hat{m}_k = \mu \{S - \hat{n}_k\}, \\ \ll \hat{M} \gg &\equiv \bar{M} = -\partial F / \partial \mathcal{H} = -\partial E_0 / \partial \mathcal{H} - \sum_k (\partial F^{01} / \partial E_k^0) (\partial E_k^0 / \partial \mathcal{H}) \\ &= \mu R \{S - \bar{n}^{01}\}, \\ \ll \hat{m} \gg &\equiv \bar{m} = \mu \{S - \bar{n}^{01}\} = \bar{m}^0 + \bar{m}^1 \equiv \bar{m}^{01}, \\ \bar{m}^0 &= \mu \{S - \bar{n}^0\} \quad \bar{m}^1 = -\mu \bar{n}^1\end{aligned}\quad (39)$$

or from the general formula given in [36] which, when specified⁶ for the Hamiltonian (19) and — consistently — for the approximation (33) reads

$$\ll Q \gg_{01} = \ll Q \gg_0 [1 + \ll H_1 \gg_0 / KT] - \ll Q H_1 \gg_0 / KT \equiv \bar{Q}^0 + \bar{Q}^1 \equiv \bar{Q}^{01} \quad (40)$$

where Q is any operator which commutes with H_0 to a c -number, $\ll Q \gg_{01}$ is its statistical average correct to first order, and $\ll \gg_0$ denotes zeroth-order averages, *i. e.*, for $H = H_0$ in Eq. (37). (For the case $[Q, H_0] \neq c$ -number and/or if approximation (33) is not made, see [36].) Putting $Q = \hat{n}$ one easily proves that

$$\begin{aligned}\ll H_1 \gg_0 &= R^{-1} \sum_{kk'} D_{kk'} \bar{n}_k^0 \bar{n}_{k'}^0, \\ \ll \hat{n} H_1 \gg_0 &= R^{-1} \sum_k \ll \hat{n}_k H_1 \gg_0, \\ \ll \hat{n}_k H_1 \gg_0 &= \bar{n}_k^0 R^{-1} \sum_{k'} \bar{n}_{k'}^0 \{2(\bar{n}_k^0 + 1) D_{kk'} + \sum_{k''} n_{k''}^0 D_{k'k''}\}\end{aligned}\quad (41)$$

with \bar{n}_k^0 from Eq. (30) and \hat{n} , \hat{n}_k as defined in (38). Hence, we obtain for \bar{n} and \bar{m} , correct to first order,

$$\begin{aligned}\bar{n}^0 &= R^{-1} \sum_k \bar{n}_k^0, \quad \bar{n}^1 = R^{-1} \sum_k \bar{n}_k^1, \\ \bar{n}_k^1 &= -(2/RKT) \bar{n}_k^0 (\bar{n}_k^0 + 1) \sum_{k'} D_{kk'} \bar{n}_{k'}^0,\end{aligned}\quad (42)$$

$$\bar{m}_0 = \mu(S - \bar{n}^0), \quad \bar{m}^1 = -\mu \bar{n}^1. \quad (43)$$

⁶ *I. e.*, put $M = m = 0$, $H_M = E_0 + H_0$, $H_{M+1} = H_1$, and $Q = \hat{n}$ or \hat{m} in Eq. (22) in [36], and note that $[\hat{n}, H_0] = [\hat{m}, H_0] = 0$ as $[\hat{n}_k, H_0] = 0$.

E.g., for the nearest -neighbour approximation in cubic lattices we have $D_{kk'} \leq 0$ and hence $\bar{n}^1 > 0$, $\bar{m}^1 < 0$.

The results (42), (43) are obtained without any reference to the system's free energy. Obviously, they must agree with those following from formulae (38), (39). This provides a check on the correctness of the free energy. Actually, one easily proves that $\bar{n}_k^1 = \partial F^1 / \partial E_k^0$ calculated from (30) is in agreement with \bar{n}_k^1 from (42), thus \bar{n}^1 and \bar{m}^1 also coincide. This completes the indirect proof that F as given by (30) is correct despite the incorrect derivation of \bar{E}^1 . Direct proofs are given in Sec. 6 and in Part II.

5. The statistical perturbation method [32]

Wallace's statistical perturbation method is designed to calculate statistical averages without having to compute the free energy. The basic requirement is to find a suitable set of operators A_n, A_n^+ that satisfy the Hamiltonian commutator equations

$$[H, A_n^+] = c_n A_n^+ + R_n^+ \quad (44)$$

where c_n are real, positive numbers and the remainder operator R_n^+ is to be small, in the sense that statistical averages involving R_n^+ are to be small. Although Wallace derives a quite general formula for the statistical average of an arbitrary operator, the formula is truly efficient when working with canonically adjoint operators. This is plainly seen in [33] where Wallace applies this method to the Heisenberg ferromagnet and uses Fourier transforms of spin operators. We shall show that the same results for unrenormalized spin waves can be obtained much easier with our Hamiltonian (19).

The operator R_k^+ follows from the equation

$$\begin{aligned} [H, b_k^+] &= E_k^0 b_k^+ + R_k^+, \\ R_k^+ &= R^{-1} \sum_{k', k'', k'''} G_{kk'k''k'''} \delta_{k'+k'', k+k'''} b_{k'}^+ b_{k''}^+ b_{k'''}^+, \\ G_{kk'k''k'''} &= A_{k-k'} + A_{k'-k''} - A_k - \frac{1}{2} A_k - \frac{1}{2} A_{k'''} \end{aligned} \quad (45)$$

According to Eq. (2.33) in [32], we have for \bar{n}_k^{01} the formula (correct to first order)

$$\bar{n}_k^{01} = \ll b_k^+ b_k \gg_{01} = \bar{n}_k^0 - (\bar{n}_k^0 + 1) \ll R_k^+ b_k \gg_0 / KT. \quad (46)$$

Since

$$\ll b_k^+ b_k^+ b_{k'} b_{k''} \gg_0 = \bar{n}_k^0 \bar{n}_{k'}^0 \{ \delta_{kk'} \delta_{k''} + \delta_{kk''} \delta_{k'} \}, \quad (47)$$

we easily obtain

$$\ll R_k^+ b_k \gg_0 = 2\bar{n}_k^0 R^{-1} \sum_{k'} D_{kk'} \bar{n}_{k'}^0. \quad (48)$$

Hence

$$\bar{n}_k^{01} = \bar{n}_k^0 - (2/RKT) \bar{n}_k^0 (\bar{n}_k^0 + 1) \sum_{k'} D_{kk'} \bar{n}_{k'}^0 = \bar{n}_k^0 + \bar{n}_k^1 \quad (49)$$

which is precisely the result (42). If we are not interested in the free energy (which takes quite a lot of calculations to be obtained by this method; see [33]), this is evidently the simplest way to calculate the magnetization.

As shown in [32], one can also calculate by this method the statistical averages \bar{E}_k^1 of the first-order corrections E_k^1 to the single-particle energies E_k^0 , by introducing into (45) renormalization parameters \tilde{E}_k^1 , i.e.,

$$[H, b_k^+] = (E_k^0 + \tilde{E}_k^1)b_k^+ + \tilde{R}_k^+, \quad \tilde{R}_k^+ = R_k^+ - \tilde{E}_k^1 b_k^+, \quad (50)$$

to be determined from the condition

$$\ll \tilde{R}_k^+ b_k \gg_0 = 0 \quad (51)$$

which leads to

$$\tilde{E}_k^1 = \ll R_k^+ b_k \gg_0 / \bar{n}_k^0 = 2R^{-1} \sum_{k'} D_{kk'} \bar{n}_{k'}^0, \quad (52)$$

due to (48). Comparison of the above result with the zeroth-order statistical average \bar{E}_k^1 obtained from (25) by replacing n_k by \bar{n}_k^0 shows that these corrections differ:

$$\tilde{E}_k^1 = \bar{E}_k^1 + 2R^{-1}(A_k - A_0)\bar{n}_k^0. \quad (53)$$

The average first-order correction \tilde{E}^1 to an energy level of the system is

$$\tilde{E}^1 = \tilde{\sum}_k \tilde{E}_k^1 = R^{-1} \sum_{kk'} D_{kk'} \bar{n}_k^0 \bar{n}_{k'}^0 (1 + \delta_{kk'}) \quad (54)$$

where the tilde over the sum means that rule (27) is to be applied. This result differs from \bar{E}^1 as well as \bar{E}^1 from Eq. (35). We thus have three different results for the average first-order energy correction to choose from, of which \bar{E}^1 should actually be least relied upon because of its incorrect derivation (violation of rule (34)). Ironically, it is precisely the latter that yields the correct first-order contribution to the free energy. This inconsistency is shown in Part II to disappear when using the standard perturbation method.

6. The standard statistical perturbation method [36]

Actually, the standard statistical perturbation procedure has already been applied in Sec. 4, Eq. (40), in calculating statistical averages. Here we shall show that the free energy F , correct to first order, can equally easily be determined. By putting $M = m = 0$ in the general formula (11) in [36] we obtain

$$F_{01} = F_0 - KT \ln [1 - \ll H_1 \gg_0 / KT]. \quad (55)$$

From (19), (30) and (38) we have $F_0 = E_0 + F^0$, $\ll H_1 \gg_0 = F^1$. Hence, when adopting the restriction (33) we obtain for (55)

$$F_{01} = E_0 + F^0 + F^1 \quad (56)$$

which coincides with (30). This is the direct proof that (30) is correct.

We would like to emphasize the simplicity of the standard statistical perturbation method which actually resides in two formulae: for the free energy and for statistical averages — the latter being superfluous if the average can be obtained from the free energy (which is mostly the case). In our case, the entire thermodynamics (of practical interest) of the Heisen-

berg ferromagnet — correct to first order and equivalent to Dyson's first Born approximation — is comprised in the simple formula (55) or, if the restriction (33) is adopted, in the simpler still formula (56). These formulae hold for any Hamiltonian, provided H_1 is small in comparison with H_0 .

Analogous investigations for the antiferromagnetic, ferrimagnetic and antiferrimagnetic Heisenberg spin Hamiltonians with uniaxial anisotropy are under way and will be published in separate papers [38].

REFERENCES

- [1] F. Bloch, *Z. Phys.*, **61**, 206 (1930); *ibid.*, **74**, 295 (1932).
 [2] C. Moeller, *Z. Phys.*, **82**, 559 (1933).
 [3] F. Keffer, *Spin Waves*, in: *Handbuch der Physik*, Vol. XVII, Springer-Verlag, Berlin—New York 1966.
 [4] A. I. Akhiezer, V. G. Baryakhtar and S. V. Peletinsky, *Spin Waves*, Moscow 1967 (in Russian).
 [5] S. V. Tyablikov, *Methods of the Quantum Theory of Magnetism*, Moscow 1965 (in Russian).
 [6] S. Szczeniowski, J. Morkowski and J. Szaniecki, *Phys. Status Solidi*, **3**, 537 (1963).
 [7] A. I. Akhiezer, V. G. Baryakhtar and M. J. Kaganov, *Uspekhi Fiz. Nauk* (USSR), **71**, 533 (1960); *ibid.*, **72**, 3 (1960).
 [8] J. van Kranendonk and J. H. van Vleck, *Rev. Mod. Phys.*, **30**, 1 (1958).
 [9] T. Holstein and H. Primakoff, *Phys. Rev.*, **58**, 1098 (1940).
 [10] R. Kubo, *Rev. Mod. Phys.*, **25**, 344 (1953).
 [11] W. Marshall, *Proc. Roy. Soc. A* (London), **232**, 69 (1955).
 [12] F. J. Dyson, *Phys. Rev.*, **102**, 1217 a. 1230 (1956).
 [13] S. V. Maleyev, *Zh. Eksper. Teor. Fiz.* (USSR), **33**, 1010 (1957).
 [14] T. Morita, *Progr. Theor. Phys.*, **20**, 614 (1958).
 [15] S. H. Charap and R. P. Weiss, *Phys. Rev.*, **116**, 1372 (1959).
 [16] T. Oguchi, *Phys. Rev.*, **171**, 117 (1960).
 [17] J. Morkowski, *Acta Phys. Polon.*, **19**, 3 (1960).
 [18] L. Kowalewski, *Acta Phys. Polon.*, **19**, 59 (1960).
 [19] F. Keffer and R. Loudon, *J. Appl. Phys. Suppl.*, **32**, 2 S (1961).
 [20] R. A. Tahir-Kheli and D. ter Haar, *Phys. Rev.*, **127**, 95 (1962).
 [21] M. Bloch, *Phys. Rev. Letters*, **9**, 286 (1962).
 [22] G. G. E. Low, *Proc. Phys. Soc. A* (London), **79**, 473 (1962).
 [23] J. Szaniecki, *Acta Phys. Polon.*, **22**, 379 a. 389 (1962); *Phys. Rev.*, **129**, 1018 (1963).
 [24] M. Wortis, *Phys. Rev.*, **138**, 1126 A (1965).
 [25] S. T. Dembiński, *Physica*, **30**, 1217 (1964).
 [26] S. H. Charap, *Phys. Rev.*, **136**, 1131 A (1964).
 [27] T. Morita and T. Tanaka, *Phys. Rev.*, **137**, 648 A (1965); **138**, 1395 A a. 1403 A (1965); *J. Math. Phys.*, **6**, 1152 (1965).
 [28] P. D. Loly and S. Doniach, *Phys. Rev.*, **144**, 319 (1966).
 [29] A. Pękalski and W. J. Ziętek, *Acta Phys. Polon.*, **30**, 839 (1966); **31**, 131 (1967).
 [30] R. Brout and F. Englert, *Bull. Amer. Phys. Soc.*, **6**, 55 (1961).
 [31] D. C. Wallace, *Phys. Rev.*, **152**, 247 (1966).
 [32] D. C. Wallace, *Phys. Rev.*, **152**, 261 (1966).
 [33] D. C. Wallace, *Phys. Rev.*, **153**, 547 (1967).
 [34] S. V. Tyablikov, *Ukr. Math. Zh.* (USSR), **11**, 287 (1959).
 [35] H. B. Callen, *Phys. Rev.*, **130**, 890 (1963).
 [36] W. J. Ziętek, *Acta Phys. Polon.*, **33**, 449 (1968).
 [37] N. N. Bogolyubov, *Lectures on Quantum Statistics*, Kiev 1949 (in Ukrainian).
 [38] G. Kozłowski, S. Krzemiński, A. Pękalski — a series of papers to be published in *Acta Phys. Polon.* (1969).