ASYMPTOTIC CONDITION IN QUANTUM THEORY OF A FIELD WITH A PARAMETER

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Under usual axioms and some additional assumptions the asymptotic condition for a field $A(x, s)$ with a parameter is proved. In addition to the results analogous to the one of Hepp in the Haag-Ruelle theory some new theorems are derived. The possibility of replacing the non-overlapping states by usual scattering states is discussed.

1. Introduction

The aim of this paper is the investigation of the asympotic condition of a field $A(x, s)$ where s is a continuous parameter. This problem was formulated and partially solved by Licht [1]. The asymptotic condition for such a field was proved [2] in a particular case, namely for the Zachariasen-Thirring model [3]. The proof was given in analogy to the approach of Hepp [4] for the Araki-Haag-Ruelle theory [5], [6], [7]. It will be shown that this condition is fulfilled by a class of fields with a parameter, which satisfy axioms analogous to the axioms of Wightman and some additional assumptions connected with the truncated vacuum expectation values (TVEV). It seems to us that the theory of a field with a parameter is not only a mathematical construction which generalizes the well-known field theory, but can be applied to some interesting physical problems. The task of applying such fields to the scattering theory was undertaken by Lukierski in [8]. A list of axioms and the additional assumptions are given in Section 2.

A summary of results and the main principles of proofs are contained in Section 3. Some new theorems can be found there which cannot be proved in the usual field theory.

For the sake of transparentness we transferred the details of the calculations to the Appendix.

2. The axioms and assumptions

1. The space of states is the Hilbert space over the field of complex numbers.

2. We consider the mapping of the space $\mathcal{S}_{R^5}$ into the space $\{A(f, u)\}$ of linear operators acting on some set D. The set D is dense in the Hilbert space H and

$$A(f, u)\, D \subset D.$$

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We will use the notation

$$A(f, u) = \int_{\mu^2}^{\infty} ds\, u(s) \int dx f(x)\, A(x, s)$$

where

$$f \in \mathcal{S}_{R^4},\, u^* = u \in \mathcal{S},\, \text{supp}\, u = \{s : s \geqslant \mu^2 > 0\}. \tag{2.1}$$

For $\Phi, \Psi \in D$ the scalar product $(\Phi, A\,(x, s)\,\Psi)$ is a tempered distribution and $(\Phi, A(f, u)\Psi) = (A(f^*, u)\,\Phi, \Psi)$.

3. There exists a unitary, continuous representation $U(a, \Lambda)$ of the inhomogeneous, proper, orthochronous Lorentz group such that $U(a, \Lambda)\, D \subset D$ and

$$U(a, \Lambda)\, A(f, u)\, U^{-1}(a, \Lambda) = A(f_{(a,\Lambda)}, u)$$

or

$$U(a, \Lambda)\, A(f, s)\, U^{-1}(a, \Lambda) = A(f_{(a,\Lambda)}, s)$$

where

$$f_{(a,\lambda)}(x) = f(\Lambda^{-1}(x-a)).$$

4. The spectrum of the infinitesimal generators P_μ of the representation of the translation group

$$U(a, 1) = \exp\{iPa\}, \qquad Pa = P_0\, a_0 - \boldsymbol{Pa}$$

is the following:

a. The point $\{p = 0\}$ is an isolated eigenvalue and the subspace Ω belonging to it is one-dimensional. $\Omega \in D$.

b. The other part of the spectrum is contained in V_+.

5. $[A(f_1, u_1), A(f_2, u_2)]\, \Phi = 0$ for $\Phi \in D$ when $\text{supp} f_2 \sim \text{supp} f_1$.

6. Let $\mathcal{L}\,(A)$be the ring of operators $A(f, u)$ over C where the functions f and u satisfy (2.1), then $\mathcal{L}(A)\, \Omega$ is dense in H.

Notice that $A(x, u)$ is a set of fields fulfilling the usual axioms of the field theory, where u is playing the role of an index. For a fixed u the axiom 6 may be violated.

One of the well known consequences of the axioms is the asymptotic behaviour of TVEV's for large space-like separations of the arguments. Namely for any integer M there exists an integer $N = N(M)$ such that

$$\prod_{i=1}^{k} (1+(x_i^0)^2)^{-N/2} \prod_{i=1}^{k-1} (1+(x_{i+1}^j - x_i^j)^2)^{M/2} \langle A(x_1^0, \boldsymbol{x}_1, u_1) \dots A(x_k^0, \boldsymbol{x}_p, u_k)\rangle_0^{\mathrm{T}} \tag{2.2}$$

is a bounded distribution in x_i^0, $\xi_l^j = x_{l+1}^j - x_l^j$, $j = 1, 2, 3$.

From the arbitrariness of the function u_i it follows that the distribution

$$\langle A(x_1, s_1) \dots A(x_n, s_n)\rangle_0^{\mathrm{T}} \prod_{k=1}^{n} (1+s_k)^{-L}$$

has the same property.

Our considerations will require some additional assumptions for the two-point Wightman functions expressed by

$$\langle A(x, s_1)\, A(y, s_2)\rangle_0 = i\int_0^\infty d\varkappa^2\, \varrho(\varkappa^2, s_1, s_2)\, \Delta^+(x-y; \varkappa^2). \tag{2.3}$$

We will assume the following form of the distribution $\varrho(\varkappa^2, s_1, s_2)$

i.

$$\varrho(\varkappa^2, s_1, s_2) = \sigma_1(\varkappa^2)\, \delta(\varkappa^2-s_1)\, \delta(\varkappa^2-s_2)+$$
$$+\sigma_2(\varkappa^2, s_1, s_2)\left[\delta(\varkappa^2-s_1) - \frac{\alpha(\varkappa^2, s_1)}{s_1-\varkappa^2+i\varepsilon}\right]\left[\delta(\varkappa^2-s_2) - \frac{\alpha(\varkappa^2, s_2)}{s_2-\varkappa^2+i\varepsilon}\right]^* +$$
$$+\sigma_3(\varkappa^2, s_1, s_2)\, P\frac{1}{\varkappa^2-s_1}\, P\frac{1}{\varkappa^2-s_2} + \sigma_4(\varkappa^2, s_1, s_2) \tag{2.4}$$

where the function $\alpha(\varkappa^2, s)$ is continuous, polynomially bounded and has the following properties:

$$\operatorname{Im}\alpha(\varkappa^2, \varkappa^2) = \pi\alpha(\varkappa^2, \varkappa^2)\, \alpha^*(\varkappa^2, \varkappa^2) \tag{2.5}$$

and

$$\sigma_i(\varkappa^2, s_1, s_2) = \gamma_i(s_1, s_2) \sum_{j=1}^{Qi} \delta(\varkappa^2-a_j) + \Gamma_i(\varkappa^2, s_1, s_2), \quad (i = 1, 2, 3, 4)$$

also

$$\gamma_i(s_1, s_2) \in \mathcal{O}_{s_1 s_2} \text{ and } \gamma_i(s_1, s_2) = \gamma_i(s_2, s_1)$$

and

$$\Gamma_i(\varkappa^2, s_2, s_1) \in \mathcal{O}_{s_1 s_2}, \quad \Gamma_i(\varkappa^2, s_1, s_2) = \Gamma_i(\varkappa^2, s_2, s_1).$$

The functions Γ_i are piecewise continuous and of polynomial growth in the variable $\varkappa^2$. $\Gamma_i(\varkappa^2, s_1, s_2) = 0$ when $\varkappa^2 < \mu^2 > 0$. The numbers $a_i > \mu^2$.

The next assumption refers to the higher TVEV's.

ii. We will assume that $\langle A(x_1, s_1) \ldots A(x_n, s_n)\rangle_0^T$ for $n \geqslant 3$

a. is a function piecewise continuous and of polynomial growth in $s_1, \ldots, s_n$.

b. is C^∞ and polynomially bounded in $s_1, \ldots, s_n$.

3. Statement of theorems

We shall investigate the asymptotic behaviour of the operator

$$A_t(f, u) = \int_0^\infty ds\, u(s) \int dx\, f^*(x, t, s)\, A(x, s) \tag{3.1}$$

where

$$f(x, t, s) = (2\pi)^{-3/2} \int dp\, \tilde{f}(p)\, e^{i(p_0-\omega_{p,s})t}\, e^{-ipx}, \; \omega_{p,s} = \sqrt{\boldsymbol{p}^2+s} \tag{3.2}$$

for $|t|$ tending to infinity. For this purpose we shall prove the following.

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Lemma 1. Under the assumptions 2, 3, 4, 5 and *iib*

$$(1+|t|)^N \langle A_t^{(*)}(f_1, u_1) \ldots A_t^{(*)}(f_n, u_n) \rangle_0^{\mathrm{T}} \tag{3.3}$$

is bounded for all t and any integer N if $n \geqslant 3$.

Proof:
$$\langle A_t^{(*)}f_1, u_1) \ldots A_t^{(*)}(f_n, u_n) \rangle_0^{\mathrm{T}}$$

$$= \int dx_1 \ldots dx_n \int dp_1 \ldots dp_n \int ds_1 \ldots ds_n e^{i\sum_1^n (\mp \omega_{p_i s_i})t} e^{i\sum_1^n \pm \{p_i^0(t-x_i^0)+\mathbf{p}_i\mathbf{x}_i\}} \times$$

$$\times \prod_1^n f_i^{(*)}(p_i)\, u_i(s_i) \langle A(x_1, s_1) \ldots A(x_n, s_n) \rangle_0^{\mathrm{T}}$$

$$= \int dx_1 \ldots dx_n \int dp_1 \ldots dp_n \int ds_1 \ldots ds_n\, e^{i\sum_1^n (\mp \omega p_i s_i)t} e^{i\sum_1^n (\mp px)} \times$$

$$\times \langle A(x_1, s_1) \ldots A(x_n, s_n) \rangle_0^{\mathrm{T}} \prod_{i=1}^n f_i^{(*)}(p_i) u_i(s_i)$$

$$= \int pd_1 \ldots dp_n \int ds_1 \ldots ds_n\, e^{i\sum_1^n (\mp \omega p_i, s_i)t}\, \tilde{W}(p_1, \ldots, p_n; s_1, \ldots, s_n) \times$$

$$\times \prod_{i=1}^n \tilde{f}_i^{(*)}(\mp p_i) u_i(s_i). \tag{3.4}$$

According to the translation invariance

$$\tilde{W}(p_1, \ldots, p_n; s_1, \ldots, s_n) = \delta(\sum_i^n p_i)\, \hat{W}(p_2, \ldots, p_n; s_1 \ldots, s_n).$$

Because of the boundedness of the distribution

$$\langle A(x_1, s_1) \ldots A(x_n, s_n) \rangle_0^{\mathrm{T}}$$

in the arguments $\xi_i = x_{i+1} - x_i$ (see formula (2.2)), $\hat{W}(p_2, \ldots, p_n; s_1, \ldots, s_n)$ is C^∞ and polynomially bounded in $\mathbf{p}_2, \ldots, \mathbf{p}_n$ and $s_1, \ldots, s_n$ (according to the assumption *iib*), when integrated over $p_2^0, \ldots, p_n^0$ with a test function from $\mathcal{S}_{R^{n-1}}$.

Now we can express (3.4) in the form

$$\zeta(t) = \int d^3p_2 \ldots d^3p_n \int ds_1 \ldots ds_n e^{iB(\mathbf{p}_2, \ldots, \mathbf{p}_n; s_1, \ldots, s_n)t}\, \chi(\mathbf{p}_2, \ldots, \mathbf{p}_n; s_1, \ldots, s_n) \tag{3.5}$$

where

$$\chi(\mathbf{p}_2, \ldots, \mathbf{p}_n; s_1, \ldots, s_n)$$

$$= \int dp_2^0 \ldots dp_n^0 f_1^{(*)}(\pm \sum_2^n p_i) \prod_2^n \tilde{f}_j^{(*)}(\mp p_j) \prod_{k=1}^n u_k(s_k)\, \tilde{W}(\mathbf{p}_2, \ldots, \mathbf{p}_n; s_1, \ldots, s_n) \tag{3.6}$$

belongs to $\mathcal{S}$ in $s_1, \ldots, s_n$ and $\mathbf{p}_2, \ldots, \mathbf{p}_n$.

$$B(\mathbf{p}_2, \ldots, \mathbf{p}_n; s_1, \ldots, s_n) = \mp \sqrt{(\sum_2^n \mathbf{p}_i)^2 + s_1} \mp \sqrt{\mathbf{p}_2^2 + s_2} \mp \ldots \mp \sqrt{\mathbf{p}_n^2 + s_n}.$$

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Since

$$\frac{\partial B}{\partial s_1} = \mp \frac{1}{\sqrt{(\sum_i^{n'} p_i)^2 + s_1}} \neq 0$$

the transformation $B \leftrightarrow s_1$ is reversible and the function $\zeta(t) \in \mathcal{S}_{R^1}$. This proves lemma 1. Basing on this lemma we shall prove some theorems. Let us consider at the beginning the problem of asymptotic states.

Theorem 1. If $\tilde{g}_i \in \mathcal{S}$ and $\operatorname{supp} \tilde{g}_i \subset V_+$, then

$$s - \lim_{|t| \to \infty} \prod_{i=1}^{k} A_t^{(*)}(g_i, u_i)\, \Omega = \Phi^{\mathrm{ex}}, \tag{3.7}$$

where Φ^{ex} is a vector of the Fock spaace of the free Licht field.

Proof:

We shall show that

$$\left\| \frac{d}{dt} \prod_{1}^{k} A_t^{(*)}(g_i, u_i)\, \Omega \right\| \leqslant C_N (1 + |t|)^{-N}. \tag{3.8}$$

Differentiating over t and splitting into the sum of TVEV's one can see that by virtue of lemma 1 is sufficient to consider the products of the two-point Wightman functions only. Since $\operatorname{supp} \tilde{g} \subset V_+$, the functions of the type $\langle \overset{(\cdot)}{A} \overset{(\cdot)}{A} \rangle_0$, $\langle \overset{(\cdot)}{A^*} \overset{(\cdot)}{A} \rangle_0$, $\langle \overset{(\cdot)}{A^*} \overset{(\cdot)}{A^*} \rangle$ $\left(\overset{(\cdot)}{A} \text{ means } A \text{ or } \dot{A} = \frac{dA}{dt} \right)$ vanish. In connection with this we have to investigate the following expression

$$\langle \overset{(\cdot)}{A_t}(g_1, u_{i_1})\, \overset{(\cdot)}{A_t^*}(g_{i_2}, u_{i_2}) \rangle_0 \ldots \langle \overset{(\cdot)}{A_t}(g_{i_{2k-1}}, u_{i_{2k-1}}) \overset{(\cdot)}{A^*}(g_{i_{2k}}, u_{i_{2k}}) \rangle_0. \tag{3.9}$$

As shown in the Appendix (lemma 2.A) the function $\langle A_t\, A_t^* \rangle_0$ tends to a constant when $|t| \to \infty$ while the function containing at least one operator differentiated over t tends to zero for $|t| \to \infty$. This accomplishes the proof. The sets of vectors $\Phi^{\mathrm{in}}_{\mathrm{out}}$ we shall denote by $D_\mp$. The density of $D_\mp$ in H is assumed.

Theorem 2. If $\tilde{f}_i \in \mathcal{S}$, then

$$s - \lim_{|t| \to \infty} \prod_{i=1}^{k} A_t^*(f_i, u_i)\, \Omega = \Psi^{\mathrm{ex}}. \tag{3.10}$$

Proof:

In the same manner as in the proof of theorem 1 we come to the conclusion that it is enough to consider the products of functions of the type $\langle \overset{(\cdot)}{A_t} \overset{(\cdot)}{A_t} \rangle_0$, $\langle \overset{(\cdot)}{A_t^*} \overset{(\cdot)}{A_t^*} \rangle_0$, $\langle \overset{(\cdot)}{A_t} \overset{(\cdot)}{A_t^*} \rangle_0$. By virtue of lemma 1.A functions $\langle \overset{(\cdot)}{A_t} \overset{(\cdot)}{A_t} \rangle_0$ and $\langle \overset{(\cdot)}{A_t^*} \overset{(\cdot)}{A_t^*} \rangle_0$ behave like $|t|^{-3/2}$ for large $|t|$. It is easy ot see that there cannot appear a product consisting only of functions $\langle A_t\, A_t^* \rangle_0$. Then according to lemma 1.A and lemma 2.A

$$\left\| \frac{d}{dt} \prod_{i=1}^{k} A_t^*(f_i, u_i)\Omega \right\|^2 \xrightarrow[|t|\to\infty]{} 0$$

like $|t|^{-3/2}$ and theorem 2 is proved.

Theorem 3. If $f_i \in \mathcal{S}$, then

$$\lim_{|t|\to\infty} (\Phi^{\text{ex}}, \prod_{i=n+1}^{n+m} A_t(f_i, u_i) \prod_{j=1}^{n} A_t^*(f_j, u_j)\Omega)$$

$$= (\Phi^{\text{ex}} \prod_{i=n+1}^{n+m} A_{\text{ex}}(\hat{f}_i, u_i) \prod_{j=1}^{n} A_{\text{ex}}^*(\hat{f}_j, u_j)\,\Omega). \tag{3.11}$$

Proof:

In this proof one uses the lemma on the weak convergence:

If

1. $||\Phi_\alpha(t) - \Phi_\alpha^{\text{ex}}|| \to 0$,
2. $||\Psi_t|| \leqslant C$,
3. the set of vectors Φ_α^{ex} is dense in H,
4. $(\Phi_\alpha(t), \Psi_t) \xrightarrow[|t|\to\infty]{} (\Phi_\alpha^{\text{ex}}, \Psi)$,

then

$$w\text{—}\lim_{|t|\to\infty} \Psi_t = \Psi.$$

Let

$$\Phi_\alpha(t) = \prod_{i=1}^{\alpha} A_t^*(g_i, u_i)\,\Omega, \ \text{supp}\ \tilde{g}_i \subset V_+$$

and

$$\Psi_t = \prod_{i=n+1}^{n+m} A_t(f_i, u_i) \prod_{j=1}^{n} A_t^*(f_j, u_j)\,\Omega$$

Then the assumptions 1. and 3. are fulfilled. All we have to check is

$$||\prod_{i=n+1}^{n+m} A_t(f_i, u_i) \prod_{j=1}^{n} A_t^*(f_j, u_j)\,\Omega||^2 < C \text{ for all } t \tag{3.12}$$

and

$$\lim_{|s|\to\infty} (\prod_{i=1}^{\alpha} A_t^*(g_i, u_i)\,\Omega, \prod_{j=n+1}^{n+m} A_t(f_i, u_i) \prod_{k=1}^{n} A_t^*(f_k, u_k)\Omega)$$

$$= (\Phi^{\text{ex}}, \prod_{j=n+1}^{n+m} A_{\text{ex}}(f_i, u_i) \prod_{k=1}^{n} A_{\text{ex}}^*(f_k, u_k)\,\Omega). \tag{3.13}$$

Let us split (3.12) and (3.13) into TVEV's. Similarly as in the former proofs it is sufficient to consider the two-point Wightman function only. Since $||\Psi_t||$ contains functions $\langle A_t A_t \rangle_0$

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$\langle A_t^* A_t^* \rangle_0, \langle A_t^* A_t \rangle_0, \langle A_t A_t^* \rangle$ which according to lemmas 1A. and 2A. are bounded in t, therefore (3.12) is fulfilled. In order to prove (3.13) we have to consider functions $\langle A_t A_t \rangle_0$, $\langle A_t^* A_t^* \rangle_0$ and $\langle A_t A_t^* \rangle_0$. The terms which contain $\langle A_t A_t \rangle_0$ or $\langle A_t^* A_t^* \rangle_0$ vanish asymptotically while the limit of $\langle A_t A_t^* \rangle_0$ is given by (A.8). This completes the proof of (3.12) and (3.13).

Basing on the previous theorems and lemma 1 one can prove the following:

Theorem 4. If $\Psi^{\text{ex}} \in D_\pm, \tilde{g}_i, \tilde{f}_i \in \mathcal{S}$ and $\operatorname{supp} \tilde{g}_i \subset V_+$ then

$$1.\ s\text{—}\lim_{|t|\to\infty} \prod_{i=1}^{n} A_t^{(*)}(g_i, u_i)\, \Psi^{\text{ex}} = \prod_{i=1}^{n} A_{\text{ex}}^{(*)}(\hat{g}_i, u_i)\, \Psi^{\text{ex}}.$$

$$2.\ s\text{—}\lim_{|t|\to\infty} \prod_{i=1}^{n} A_t^{*}(f_i, u_i)\, \Psi^{\text{ex}} = \prod_{i=1}^{n} A_{\text{ex}}^{*}(\hat{f}_i, u_i)\, \Psi^{\text{ex}}$$

$$3.\ w\text{—}\lim_{|t|\to\infty} \prod_{i=n+1}^{n+m} A_t(f_i, u_i) \prod_{j=1}^{n} A_t^{*}(f_j, u_j)\, \Psi^{\text{ex}}$$

where the operators $A_t^{(*)}$ are extensions of the original fields defined on $\mathcal{P}(A)\Omega$.

The proof will be omitted because it is analogous to the proof of theorem 2.2 in [9].

As one can see from the quoted theorems the asymptotic field of the field $A(x, s)$ is the so called Licht field $A_{\text{ex}}(x, s)$ such that

$$A_{\text{ex}}(x, u) = \int_0^\infty ds\, u(s)\, A_{\text{ex}}(x, s)$$

is a generalized free field with the following spectral representation:

$$\langle A_{\text{ex}}(x, u_1)\, A_{\text{ex}}(y, u_2) \rangle_0$$

$$= \int_0^\infty d\varkappa^2\, [\sigma_1'(\varkappa^2, \varkappa^2, \varkappa^2) + \pi^2\, \sigma_3'(\varkappa^2, \varkappa^2, \varkappa^2)]\, u_1(\varkappa^2)\, u_2(\varkappa^2)\, \Delta^+(x-y; \varkappa^2),$$

$$\sigma_1'(\varkappa^2, \varkappa^2, \varkappa^2) = \sigma_1(\varkappa^2, \varkappa^2, \varkappa^2) + \{1 - \pi^2 |\alpha(\varkappa^2, \varkappa^2)|^2\}\, \sigma_2(\varkappa^2\, \varkappa^2, \varkappa^2), \tag{3.14}$$

$$\sigma_3'(\varkappa^2, s_1, s_2) = \sigma_3(\varkappa^2, s_1, s_2) + \sigma_2(\varkappa^2, s_1, s_2)\, \alpha(\varkappa^2, s_1)\, \alpha^*(\varkappa^2, s_2). \tag{3.15}$$

Similarly as in [9] one can check that the relation

$$A(x, s)\, \Psi^{\text{ex}} = A_{\text{ex}}(x, s)\, \Psi^{\text{ex}} + \int dy\, \Delta_{\substack{\text{red}\\ \text{adv}}}(x-y; s)\, j(y, s)\, \Psi^{\text{ex}},$$

where

$$j(x, s) = (\Box - s)\, A(x, s)$$

holds. This is an analogue of the Yang-Feldman equation.

Until now we have used the assumption *ii*b. In the second part of the Section 3 we shall investigate the consequences of the assumption *ii*a.

Lemma 2. Under the axioms 2, 3, 4, 5 and assumption *ii*a

$$(1 + |t|)^N \langle A_t^{(*)}(f_1, u_1) \ldots A_t^{(*)}(f_n, u_n) \rangle_0^{\mathrm{T}}$$

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is bounded for all t, any integer N and $n \geqslant 3$ if the supports of the functions $\{f_k, u_k\}$ have the following property. From $\boldsymbol{p}_j \in \operatorname{supp} \tilde{f}_j, \boldsymbol{p}_k \in \operatorname{supp} \tilde{f}_k, s_j \in \operatorname{supp} u_j, s_k \in \operatorname{supp} u_k$ it follows that

$$\frac{p_k^i}{p_j^i} \neq \sqrt{\frac{s_k}{s_j}} \quad (i = 1, 2, 3) \tag{3.16}$$

for $j, k = 1, ..., n$. The relation (3.16) has to hold for at least one component of $\boldsymbol{p}$.

Functions $\{f_k, u_k\}$ with the property mentioned above shall be called, after Hepp, non-overlapping functions.

The subspace $D_0 \subset D$ built up of operators A smeared out with non-overlapping functions is dense in H.

Proof:

Similarly as in the proof of lemma 1 one gets

$$\langle A_t^{(*)}(f_1, u_1) \dots A_t^{(*)}(f_n, u_n)\rangle_0^{\mathrm{T}}$$

$$= \int d^3p_2 \dots d^3p_n \int_0^\infty ds_1 \dots ds_n \, e^{iB(\boldsymbol{p}_2, ..., \boldsymbol{p}_n; s_1, ..., s_n)t} \chi(\boldsymbol{p}_2, ..., \boldsymbol{p}_n; s_1, ..., s_n)$$

where $\chi(\boldsymbol{p}_2, ..., \boldsymbol{p}_n; s_1, ..., s_n)$ given by formula (3.6) now decreases rapidly and is piecewise continuous in arguments $s_1, ..., s_n$ and $\chi \in \mathcal{S}$ in $\boldsymbol{p}_2, ..., \boldsymbol{p}_n$.

Since the functions are non-overlapping one has

$$\frac{\partial B}{\partial \boldsymbol{p}_2} = \mp \frac{\sum_2^n \boldsymbol{p}_i}{\sqrt{(\sum_2^n \boldsymbol{p}_i)^2 + s_2}} \mp \frac{\boldsymbol{p}_2}{\sqrt{\boldsymbol{p}_2^2 + s_2}} \neq 0$$

in the support of χ. By a suitable partition of the unity $\{\beta_i\}$ $i = 1, 2, 3$ one can make the transformation $B \leftrightarrow p_2^i$ to be regular in supp $\chi\beta_i$. By multiple partial integration one proves the lemma 2.

Lemma 3. If the axioms 2–5 and assumption *iia* are satisfied, then

$$|\langle A_t^{(*)}(f_1, u_1) \dots A_t^{(*)}(f_n, u_n)\rangle_0^{\mathrm{T}}| \leqslant C(1+|t|)^{-3/2(n-2)}. \tag{3.17}$$

Proof:

Under the same argumentation as in the proof of lemma 1 one gets according to the translational invariance, that

$$\langle A_t^{(*)}(f_1, u_1) \dots A_t^{(*)}(f_n, u_n)\rangle_0^{\mathrm{T}}$$

$$= \int dx_1, \dots dx_n \int d^3p_1 \dots d^3p_n \int_0^\infty ds_1 \dots ds_n \, \langle A(-x_1^0, x_1, s_1) \dots A(-x_n^0, x_n, s_n)\rangle_0^{\mathrm{T}} \times$$

$$\times u_1(s_1) \dots u_n(s_n) f_1'(x_1^0, \boldsymbol{p}_1) \dots f_n'(x_n^0, \boldsymbol{p}_n) \, e^{\mp i(\omega_{p_1 s_1} t - \boldsymbol{p}_1 \boldsymbol{x}_1) \mp \dots \mp i(\omega_{p_n s_n} t - \boldsymbol{p}_n \boldsymbol{x}_n)} \tag{3.18}$$

where

$$f_i'(x_i^0, \boldsymbol{p}_i) = \int dp_i^0 \tilde{f}(p_i^0, \boldsymbol{p}_i)\, e^{ip_i^0 x_i^0}.$$

From the asymptotic behaviour of TVEV's for large space-like separation one can rewrite (3.18) in the form

$$\sum_{r=1}^{R} \int dx_1, \ldots dx_n \int_0^\infty ds_1, \ldots ds_n \hat{F}_r(x_1^0, \ldots, x_n^0; \xi_1, \ldots, \xi_{n-1}; s_1, \ldots, s_n) \cdot$$

$$\cdot \prod_{i=1}^{k-1} \prod_{j=1}^{3} (1+|\xi_i^j|^2)^{-1} \prod_{i=1}^{k} u_i(s_i) \int d^3p_i f_i^r(x_i^0, \boldsymbol{p}_i)\, e^{\pm i(\omega_{p_i s_i} t - \boldsymbol{p}_i \boldsymbol{x}_i)} \tag{3.19}$$

(see [4], proof of the lemma 1.1).

The functions $\hat{F}_r$ is continuous and bounded in $x_1^0, \ldots, x_n^0, \xi_1, \ldots, \xi_{n-1}$, piecewise continuous and of slow increase in $s_1, \ldots, s_n$.

Functions $f_i^r(x_i^0, \boldsymbol{p}_i) \in \mathcal{S}_{R^4}$. From the asymptotic behaviour of the solutions of the Klein-Gordon equation it follows that

$$\max_{\boldsymbol{x}_i} \left| \int d^3p_i f_i^r(x_i^0, \boldsymbol{p}_i)\, e^{\pm i(\omega_{p_i s_i} t - \boldsymbol{p}_i \boldsymbol{x}_i)} \right| \leqslant C(x_i^0, s_i)\, (1+|t|)^{-3/2}$$

where $C(x_i^0, s_i) \in \mathcal{S}_{x_i^0}$ and is bounded in s_i for $s_i \geqslant \mu^2 > 0$.

Therefore we can majorize the modulus of (3.19) by:

$$\sum_{r=1}^{R} C_r \int dx_n \left| \int_0^\infty ds_n u_n(s_n) \int d^3p_n f_n^r(x_n^0, \boldsymbol{p}_n)\, e^{\pm i(\omega_{p_n s_n} t - \boldsymbol{p}_n \boldsymbol{x}_n)} \right| (1+|t|)^{-3/2(n-1)} \leqslant Q(1+|t|)^{-3/2(n-2)}.$$

In order to prove the theorems 1, 2 and 3 it is sufficient to use TVEV's with the property proved in lemma 1 or lemma 2. Therefore if we assume *iia* the theorem 4 can be proved only for vectors $\Psi^{\mathrm{ex}} \in D_0$.

4. *Final remarks*

The result we received in the proceding Section are similar to the results obtained in the usual field theory. Theorem 1 is analogous to theorem 1.1 in [4] while theorems 2 and 3 correspond to corollary 1.1 in the same work. The statement parallel to our theorem 2 contain weak limits instead of strong ones, while the statement parallel to theorem 3 contains operators A^* (g, t) where $\tilde{g}$ has some restriction on their support.

As we have seen in section 3 it is possible to omit the introduction of non-overlapping functions by assuming *iib*. We do not know, however, what the physical sense of this assumption is and to what extent does it trivialize the theory.

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An example of a field which fulfils axioms 2–5 and the assumption *ii*b is the Zachariasen--Thirring model. In this model TVEV's of higher order are equal to zero and σ_i functions are given by the formulae:

$$\sigma_1(\varkappa^2) = 1,\ \sigma_2(\varkappa^2, s_1, s_2) = 1,\ \sigma_3(\varkappa^2, s_1, s_2) = 0$$

$$\sigma_4(\varkappa^2, s_1, s_2) = \frac{g_0^2 a_0^2 f(s_1) f(s_2)}{(s_1-\varkappa^2)(s_2-\varkappa^2)}\, \delta(\varkappa^2-\mu^2)$$

$$\alpha(\varkappa^2, s) = \frac{g_0^2 f(\varkappa^2)\, D_-(\varkappa^2) f(s)}{|D_-(\varkappa^2)|^2\, (\varkappa^2-s_0)}$$

g_0, a_0, s_0 are constants and $D_-(\varkappa^2), f(s)$ are functions characterizing this model and are defined in [3]. Finally we want to remark that if one ignores the assumption (2.5), the asymptotic condition can still be proved, but the spectral function $\varrho_{ex}\,(\varkappa^2, s_1, s_2)$ of A_{in} and A_{out} will be different and this leads to a non-unitary S-matrix.

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APPENDIX

We collect here the formulae of the two-point functions which appear in the proofs of the theorems in Section 3.

Each of the function can be written in the form

$$\int_0^\infty d\varkappa^2 \int_0^\infty ds_1\, ds_2\, u_1(s_1)\, u_2(s_2)\, \varrho(\varkappa^2, s_1, s_2)\, F_t(\varkappa^2, s_1, s_2) \tag{A.1}$$

where $F_t(\varkappa^2, s_1, s_2)$ is different for different functions. Table I gives the two-point functions with the corresponding F_t functions.

Lemma 1.A. If $G(f_1, u_1, f_2, u_2, t)$ is one of the functions 1–8 given in the Table I, then

$$G(f_1, u_1, f_2, u_2, t)\, (1+|t|)^{3/2} < C < \infty.$$

Proof:
Let us write (2.4) in the following form

$$\varrho(\varkappa^2, s_1, s_2) = \sigma_1'(\varkappa^2)\, \delta(\varkappa^2-s_1)\, \delta(\varkappa^2-s_2)-$$

$$-\sigma_2(\varkappa^2, s_1, s_2) \left[\delta(\varkappa^2-s_1)\, P\frac{\mathrm{Re}\,\alpha(\varkappa^2, s_2)}{\varkappa^2-s_2} + \delta(\varkappa^2-s_2)\, P\frac{\mathrm{Re}\,\alpha(\varkappa^2, s_1)}{\varkappa^2-s_1}\right] +$$

$$+\sigma_3'(\varkappa^2, s_1, s_2)\, P\frac{1}{\varkappa^2-s_1}\, P\frac{1}{\varkappa^2-s_2} + \sigma_4(\varkappa^2, s_1, s_2) \tag{A.2}$$

where $\sigma_1'(\varkappa^2)$ and $\sigma_3'(\varkappa^2, s_1, s_2)$ are given by (3.14) and (3.15).

TABLE I

No	Two-point function	$F_t(\varkappa^2, s_1, s_2)$
1.	$\langle A_t(f_1, u_1)A_t(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})\tilde{f}_2^*(-\mathbf{p}, -\omega_{p,\varkappa^2})e^{i\omega_{p,s_1}+\omega_{p,s_2})t}$
2.	$\langle A_t(f_1, u_2)\dot{A}_t(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})\tilde{f}_1^*(-\mathbf{p}, -\omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}+\omega_{p,s_2})e^{i(\omega_{p,s_1}+\omega_{p,s_2})t}$
3.	$\dot{A}_t^*(f_1, u_1)A_t^*(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1(-\mathbf{p}, -\omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}+\omega_{p,s_1})\, e^{-i(\omega_{p,s_1}+\omega_{p,s_1})t}$
4.	$\langle A_t^*(f_1, u_1)A_t^*(f_2, u_2)\rangle_0$	$\int \frac{d^3\mathbf{p}}{2\omega_{p,\varkappa^2}} \tilde{f}_1(-\mathbf{p}, -\omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})e^{-i(\omega_{p,s_1}+\omega_{p,s_2})t}$
5.	$\langle A_t^*(f_1, u_1)\dot{A}_t^*(\tilde{f}_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1(-\mathbf{p}, -\omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})e^{-i(\omega_{p,s_1}+\omega_{p,s_2})t}(\omega_{p,\varkappa^2}-\omega_{p,s_2})$
6.	$\langle \dot{A}_t(f_1, u_1)A_t(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})\tilde{f}_2^*(-\mathbf{p}, -\omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}-\omega_{p,s_1})e^{i(\omega_{p,s_1}+\omega_{p,s_2})t}$
7.	$\langle \dot{A}_t(f_1, u_1)\dot{A}_t(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})f_2(-\mathbf{p}, -\omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}-\omega_{p,s_1})(\omega_{p,\varkappa^2}+$ $+\omega_{p,s_2})e^{i(\omega_{p,s_1}+\omega_{p,s_2})t}$
8.	$\langle \dot{A}_t^*(f_1, u_1)\dot{A}_t^*(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1(-\mathbf{p}, -\omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}+\omega_{p,s_1})(\omega_{p,\varkappa^2}-$ $-\omega_{p,s_2})\, e^{-i(\omega_{p,s_1}+\omega_{p,s_2})t}$
9.	$\langle A_t(f_1, u_1)A_1^*(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})f_2(\mathbf{p}, \omega_{p,\varkappa^2})e^{-i(\omega_{p,s_1}-\omega_{p,s_2})t}$
10.	$\langle A_t(f_1, u_1)\dot{A}_t^*(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}-\omega_{p,s_2})e^{-i(\omega_{p,s_1}-\omega_{p,s_2})t}$
11.	$\langle \dot{A}_t(f_1, u_1)A_t^*(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}-\omega_{p,s_1})e^{-i(\omega_{p,s_1}-\omega_{p,s_2})t}$
12.	$\langle \dot{A}_t(f_1, u_1)\dot{A}_t^*(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\mathbf{p}, \omega_{p,\varkappa^2})\tilde{f}_2(\mathbf{p}, \omega_{p,\varkappa^2})(\omega_{p,\varkappa^2}-\omega_{p,s_1})(\omega_{p,\varkappa^2}-$ $-\omega_{p,s_2})e^{-i(\omega_{p,s_1}-\omega_{p,s_2})t}$
13.	$\langle A_t^*(f_1, u_1)A_t(f_2, u_2)\rangle_0$	$\int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1(-\mathbf{p}, -\omega_{p,\varkappa^2})\tilde{f}_2^*(-\mathbf{p}, -\omega_{p,\varkappa^2})e^{-i(\omega_{p,s_1}-\omega_{p,s_2})t}$

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The proof will be exemplified on the function $\langle A_t(f_1, u_1)\, A_t(f_2, u_2)\rangle_0$. If we put (A.2) into the form of $\langle A_t A_t\rangle_0$ we get a sum of the following expressions

$$\int_0^\infty d\varkappa^2 \sigma_1'(\varkappa^2) u_1(\varkappa^2) u_2(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2^*(-\boldsymbol{p}, -\omega_p\, \varkappa^2) e^{2i\omega_{p,\varkappa^2} t} \tag{A.3}$$

$$-\int_0^\infty d\varkappa^2 u_1(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2^*(-\boldsymbol{p}, -\omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_2 u_2(s_2) \operatorname{Re}[\alpha(\varkappa^2, s_2)]\, P \frac{1}{\varkappa^2 - s_2} \sigma_2(\varkappa^2, \varkappa^2, s_2)\, e^{i(\omega_{p,\varkappa^2} + \omega_{p,s_2})t} \tag{A.4}$$

$$-\int_0^\infty d\varkappa^2 u_2(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_1^*(-\boldsymbol{p}, -\omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_1\, u_1(s_1) \operatorname{Re}[\alpha(\varkappa^2, s_1)]\, P \frac{1}{\varkappa^2 - s_1} \sigma_2(\varkappa^2, s_1, \varkappa^2)\, e^{i(\omega_{p,s_1} + \omega_{p,\varkappa^2})t} \tag{A.4'}$$

$$\int_0^\infty d\varkappa^2 \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2^*(-\boldsymbol{p}, -\omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_1\, u_1(s_1)\, P \frac{1}{\varkappa^2 - s_1} \int_0^\infty ds_2 u_2(s_2)\, P \frac{1}{\varkappa^2 - s_2} \sigma_3(\varkappa^2, s_1, s_2)\, e^{i(\omega_{p,s_1} + \omega_{p,s_2})t} \tag{A.5}$$

$$\int_0^\infty d\varkappa^2 \int_0^\infty \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_1^*(-\boldsymbol{p}, -\omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_1\, u_1(s_1) \int_0^\infty ds_2\, u_2(s_2)\, \sigma_4(\varkappa^2, s_1, s_2)\, e^{i(\omega_{p,s_1} + \omega_{p,s_2})t}. \tag{A.6}$$

We will start with the investigation of the expression (A.5). For this purpose we will write (A.5) in the form

$$\lim_{t_1 \to t} \int_0^\infty d\varkappa^2 \int_0^\infty dr\, r^2\, g(r, \varkappa^2, t_1)\, e^{2i\omega_{r,\varkappa^2} t} \tag{A.7}$$

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where

$$g(r^2, \varkappa^2, t_1) = \frac{1}{\omega_{r,\varkappa^2}} \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\theta \sin\theta \tilde{f}_1^*(r, \theta\, \varphi, \omega_{r\varkappa^2}) \tilde{f}_2^*(r, \theta+\pi, \varphi, -\omega_{r\varkappa^2}) \cdot$$

$$\int\limits_0^{\infty} ds_1 P \frac{1}{s_1-\varkappa^2} u_1(s_1) e^{i(\omega_{r,s_1}-\omega_{r,\varkappa^2})t_1} \int\limits_0^{\infty} ds_2 P \frac{1}{s_2-\varkappa^2} u_2(s_2) e^{i(\omega_{r,s_2}-\omega_{r,\varkappa^2})t_1} \sigma_3'(\varkappa^2, s_1, s_2) \cdot e^{2i\omega_{r,\varkappa^2}t_1}.$$

belongs to $\mathcal{S}$ in r and is bounded an C^∞ in t_1.

Since

$$\frac{\partial}{\partial r}\left[\int\limits_0^{\infty} ds\, P \frac{1}{s-\varkappa^2} u\,(s)\, e^{i(\omega_{r,s}-\omega_{r,\varkappa^2})t}\right]$$

$$= -\, it \int\limits_0^{\infty} ds\, \frac{u(s)}{(\omega_{r,\varkappa^2}+\omega_{r,s})\omega_{r,s}\,\omega_{r,\varkappa^2}}\, e^{i(\omega_{r,s}-\omega_{r,\varkappa^2})t}$$

tends rapidly to zero when $|t| \to \infty$ one can conclude that

$$\frac{\partial^n g(\varkappa^2, r, t_1)}{\partial r^n}$$

has the same properties as $g(\varkappa^2, r, t_1)$.

By introducing a function $\nu(\boldsymbol{r}) \in C^\infty$ such that

$$\nu(r) = \begin{cases} 1 \text{ for } 0 \leqslant r < \eta_1 > 0 \\ 0 \text{ for } r > \eta_2 > \eta_1 \end{cases}$$

one can divide the integration over r in the formula (A.7) into two integrals: $\int\limits_0^{\eta_2} + \int\limits_{\eta_1}^{\infty}$.

To the first integral we will apply the stationary phase method [11] according to which we have

$$\int\limits_0^{\eta_2} g(r, \varkappa^2, t_1)\, e^{2it\omega_{r,\varkappa^2}} r^2\, \nu(r)\, dr = \sum_{n=0}^{N-1} \frac{1}{n!\,2} \frac{\partial^n k(0, \varkappa^2, t_1)}{\partial u^n} \times$$

$$\times\, \Gamma\left(\frac{(n+3)}{2}\right) e^{i\pi}(n+3)/4t^-(n+3)/2e^{2it\varkappa} + R_N(t_1, \varkappa^2, t).$$

$$|R_N(t_1, \varkappa^2, t)| \leqslant \frac{1}{(N-1)!}\, \Gamma\left(\frac{N}{2}\right) t^{-N/2} \int\limits_0^{u_1} du u^2 \left|\frac{d^N(\nu_1 k)}{du^N}\right|$$

where

$$u^2 = 2(\omega_{r,\varkappa^2} - \varkappa),\ u_1^2 = 2(\omega_{r,\varkappa^2} - \varkappa)$$

$$\nu_1(u) = \nu(r)$$

$$k(u) = g(r, \varkappa^2, t_1)\, r^2 u^{-2} \frac{dr}{du}.$$

Taking $N = 3$ one can majorize the modulus of (A.5) by

$$v_1(t_1, \varkappa^2)|t|^{-3/2} + v_2(t_1, \varkappa^2)|t|^{-5/2} + v_3(t_1, \varkappa^2)|t|^{-7/2}$$

where v_i $(i = 1, 2, 3)$ are bounded in t_1. Taking into account the properties of σ_i in $\varkappa^2$ we obtain that

$$\lim_{|t|\to\infty} \lim_{t_1\to t} \int_0^\infty d\varkappa^2 \int_0^{\eta^2} dr r^2 g(r, \varkappa^2, t_1) e^{2i\omega_{r,\varkappa^2} t} \nu(r) = 0.$$

By multiple partial integration and because of the boundedness of $\frac{\partial^n g}{\partial r^n}$ in t_1 one gets

$$\lim_{|t|\to\infty} \lim_{t_1\to t} \int_0^\infty d\varkappa^2 \int_{\eta_1}^\infty dr r^2 g(r, \varkappa^2, t_1)\, e^{2i\omega_{r,\varkappa^2} t}(1-\nu(r)) = 0.$$

The asymptotic vanishing of the expressions (A.4) and (A.4′) can be proved by using the same method as for (A.5). The expression (A.3) tends to zero like $|t|^{-3/2}$ (see [10]), while (A.6) tends to zero like $|t|^{-N}$, for any integer N.

Lemma 2.A

a. $$\lim_{|t|\to\infty} \langle A_t(f_1, u_1)\, A_t^*(f_2, u_2)\rangle_0$$

$$= -\int_0^\infty d\varkappa^2\, [\sigma_1'(\varkappa^2, \varkappa^2, \varkappa^2) + \pi\sigma_3'(\varkappa^2, \varkappa^2, \varkappa^2)]\, u_1(\varkappa^2)\, u_2(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2(\boldsymbol{p}, \omega_{p,\varkappa^2}). \tag{A.8}$$

b. $$\lim_{|t|\to\infty} \langle A_t^*(f_1, u_1)\, A_t(f_2, u_2)\rangle_0$$

$$= \int_0^\infty d\varkappa^2\, [\sigma_1' + \pi\sigma_3']\, u_1(\varkappa^2)\, u_2(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1(-\boldsymbol{p}, -\omega_{p,\varkappa^2}) \tilde{f}_2^*(-\boldsymbol{p}, -\omega_{p,\varkappa^2}) \tag{A.9}$$

c. $$\lim_{|t|\to\infty} |t^N \langle A_t(f_1, u_1)\, A_t^*(f_2, u_2)\rangle_0| = \lim_{|t|\to\infty} |t^N \langle \dot{A}_t(f_1, u_1)\, A_t^*(f_2, u_2)\rangle_0| =$$

$$= \lim_{|t|\to\infty} |t^N \langle \dot{A}_t(f_1, u_1)\, \dot{A}_t^*(f_2, u_2)| = 0 \tag{A.10}$$

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Proof of a. $\lim_{|t|\to\infty} \langle A_t(f_1, u_1)\, A_t^*(f_2, u_2)$

$$= \int_0^\infty d\varkappa^2 \sigma_1'(\varkappa^2) u_1(\varkappa^2) u_2(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}(\boldsymbol{p}, \omega_{p,\varkappa^2}) -$$

$$- \int_0^\infty d\varkappa^2 u_1(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^* (\boldsymbol{p}, \omega_{p,\varkappa^2}) f_2(\boldsymbol{p}, \omega_{p,\varkappa_2}) \times$$

$$\times \int_0^\infty ds_2 u_2(s_2)\, \sigma_2(\varkappa^2, \varkappa^2, s_2)\, \mathrm{Re}\,[\alpha(\varkappa^2, s_2)]\, P \frac{1}{\varkappa^2 - s_2}\, e^{-i(\omega_{p,\varkappa^2} - \omega_{p,s_2})t} -$$

$$- \int_0^\infty d\varkappa^2 u_2(\varkappa^2) \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2(\boldsymbol{p}, \omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_1 u_1(s_1) \sigma_2(\varkappa^2, s_1, \varkappa^2)\, \mathrm{Re}\,[\alpha(\varkappa^2, s_1|)]\, P \frac{1}{\varkappa^2 - s_1}\, e^{-i(\omega_{p,s_1} - \omega_{p,\varkappa^2})t} +$$

$$+ \int_0^\infty d\varkappa^2 \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2(\boldsymbol{p}, \omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_1 u_1(s_1)\, P \frac{1}{\varkappa^2 - s_1}\, e^{-i(\omega_{p,s_1} - \omega_{p,\varkappa^2})t} \int_0^\infty ds_2 u_2(s_2)\, \sigma_3(\varkappa^2\, s_1, s_2)\, e^{-i(\omega_{p,\varkappa^2} - \omega_{p,s_2})t} +$$

$$+ \int_0^\infty d\varkappa^2 \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2(\boldsymbol{p}, \omega_{p,\varkappa^2}) \times$$

$$\times \int_0^\infty ds_1 u_1(s_1)\, e^{-i\omega_{p,s_1}t} \int_0^\infty ds_2 u_2(s_2)\, \sigma_4(\varkappa^2, s_1, s_2)\, e^{i\omega_{ps_2}t}\,. \tag{A.11}$$

The first expression of (A.11) is constant, the second and third cancel asymptotically[1]

---

[1] One has to use the formula

$$\int_{-\infty}^{\infty} dx \varphi(x) e^{-ixt}\, P \frac{1}{x} \xrightarrow[t\to-\infty]{t\to\infty} \begin{cases} -\pi i \varphi(0) \\ \pi i \varphi(0) \end{cases}$$

The limit of the fourth term is

$$\pi^2 \int_0^\infty d\varkappa^2 u_1(\varkappa^2) u_2(\varkappa)\, \sigma_3'(\varkappa^2, \varkappa^2, \varkappa^2) \cdot \int \frac{d^3p}{2\omega_{p,\varkappa^2}} \tilde{f}_1^*(\boldsymbol{p}, \omega_{p,\varkappa^2}) \tilde{f}_2(\boldsymbol{p}, \omega_{p,\varkappa^2}).$$

The last term in (A.11) tends rapidly to zero.

The proof of b. is very similar to the proof of a.

Proof of c.

The function $\langle A_t \dot{A}_t^* \rangle_0$ has a form analogous to (A.11) with the only difference being that under the integrals the expression $(\omega_{p,\varkappa^2} - \omega_{p,s})$ appears additionally. In this case the first two terms vanish while the third and the fourth terms tend rapidly to zero, because of the appearance of

$$(\omega_{p,\varkappa^2} - \omega_{p,s})\, P \frac{1}{x^2 - s} = \frac{1}{\omega_{p,\varkappa^2} + \omega_{p,s}}\,.$$

The last term of the function $\langle A_t \dot{A}_t^* \rangle_0$ tends to zero as in the case of $\langle A_t A_t^* \rangle_0$.

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