

THE OVERLAP FUNCTIONS FOR πp INELASTIC PROCESSES AND THEIR RELATION TO PARTIAL INELASTIC CROSS-SECTIONS

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(Received November 28, 1968)

The overlap function formalism is extended to the spin $(0, \frac{1}{2})$ case. Two overlap functions for inelastic collisions — helicity non-flip and helicity flip — are introduced, the first corresponding to that from a spinless case. Their connection with partial cross-sections is derived. It is shown that the slope of the helicity non-flip overlap function practically does not depend on energy in the range from 1.5 to 8 GeV/c.

1. Introduction

In this work we discuss the overlap functions in the πp elastic scattering. The overlap function formalism was introduced by Van Hove for the scattering of spinless particles and then discussed by Henzi for πN scattering in the impact parameter approximation. The purpose of this work is to investigate the influence of the spin effects on the overlap functions. To this end we have calculated the overlap functions from the existing data and models of elastic π^+p scattering. In Section 2 we present the overlap function formalism for the πp scattering. In Section 3 the properties of the overlap functions derived from the experimental data are discussed. Our conclusions are listed in Section 4.

2. Overlap function formalism for πp scattering

The unitarity requirement for a scattering amplitude f of spin 0 particles

$$\langle f | \text{Im} f | i \rangle = \frac{k}{4\pi} \langle f | f^+ f | i \rangle \quad (1)$$

has been expressed as

$$\text{Im} f = \frac{k}{4\pi} (F_{\text{el}} + F_{\text{in}}), \quad (2)$$

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where F_{el} and F_{in} are called the overlap functions. They are connected with the partial (elastic or inelastic) cross-section σ^l by the formula:

$$F = \sum_l \sigma^l P_l(\cos \theta). \quad (3)$$

For collisions of spin 0 and 1/2 particles there are two independent unitarity relations for the helicity amplitudes

$$\langle + | \text{Im} \check{f} | + \rangle = \frac{k}{4\pi} \langle + | \check{f}^+ \check{f} | + \rangle \quad (4a)$$

$$\langle - | \text{Im} \check{f} | + \rangle = \frac{k}{4\pi} \langle - | \check{f}^+ \check{f} | + \rangle, \quad (4b)$$

where $|+\rangle$ and $|-\rangle$ denote states with helicities $+\frac{1}{2}$ and $-\frac{1}{2}$, \check{f} is the scattering matrix

$$\check{f} = \hat{I} \cdot \tilde{f}(k, \Omega) + \frac{i\vec{\sigma}(\vec{k}_f \times \vec{k}_i)}{k^2} \cdot \tilde{g}(k, \Omega) \quad (5)$$

and \vec{k}_i, \vec{k}_f are the three-momenta of the incoming and outgoing pion in the c. m. system.

Using the unity operator $\mathbf{1} = \sum_j |j\rangle \langle j|$ and separating from it the intermediate elastic states one obtains:

$$\begin{aligned} \text{Im} f_{++}^{i \rightarrow f} &= \frac{k}{4\pi} \left\{ \int d\Omega f_{++}^{i \rightarrow p} (f_{++}^{f \rightarrow p})^* + \int d\Omega f_{+-}^{i \rightarrow p} (f_{+-}^{f \rightarrow p})^* + \right. \\ &+ \left. \sum_m \langle +_f | \check{f}^+ | m \rangle \langle m | \check{f} | +_i \rangle \right\} = \frac{k}{4\pi} \{ F_{el}^{+++, ++} + F_{el}^{+-, +-} + F_{in}^{+-} \} \end{aligned} \quad (6a)$$

$$\begin{aligned} \text{Im} f_{+-}^{i \rightarrow f} &= \frac{k}{4\pi} \left\{ \int d\Omega f_{++}^{i \rightarrow p} (f_{-+}^{f \rightarrow p})^* - \int d\Omega f_{+-}^{i \rightarrow p} (f_{--}^{f \rightarrow p})^* + \right. \\ &+ \left. \sum_m \langle -_f | \check{f}^+ | m \rangle \langle m | \check{f} | +_i \rangle \right\} = \frac{k}{4\pi} \{ F_{el}^{+-, -+} + F_{el}^{+-, --} + F_{in}^{+-} \}. \end{aligned} \quad (6b)$$

Here, F_{el} are the overlap functions for elastic scattering with indices denoting the helicity of the initial, intermediate, final and again intermediate states indicating the integrated helicity amplitudes. Similarly, F_{in} are the overlap functions for inelastic channels with indices showing helicity of the initial and final two-body states, the same as in the l. h. s. of the unitarity equations. They may be called helicity non-flip and helicity flip overlap functions.

The functions f_{++}, f_{+-}, f_{-+} and f_{--} are the helicity amplitudes for the scattering of the spin 0 particles on those of spin $\frac{1}{2}$.

Their partial wave expansion is

$$f_{++} = \frac{1}{k} \sum_j \left(j + \frac{1}{2} \right) f_{++}^j D_{\frac{1}{2}\frac{1}{2}}^j(\varphi, \theta, -\varphi) = \frac{1}{k} \sum_j \left(j + \frac{1}{2} \right) f_{++}^j d_{\frac{1}{2}\frac{1}{2}}^j(\theta) \quad (7a)$$

$$f_{+-} = \frac{1}{k} \sum_j \left(j + \frac{1}{2} \right) f_{+-}^j D_{\frac{1}{2}-\frac{1}{2}}^j(\varphi, \theta, -\varphi) = \frac{1}{k} \sum_j \left(j + \frac{1}{2} \right) f_{+-}^j d_{\frac{1}{2}-\frac{1}{2}}^j(\theta) e^{i\varphi}, \quad (7b)$$

where f_{++}^j and f_{+-}^j are the partial amplitudes for the angular momentum j . We shall now derive a partial wave expansion of the overlap function F_{in}^+ , by inserting (7) into (6):

$$\begin{aligned} & \frac{1}{k} \sum_j \left(j + \frac{1}{2} \right) \text{Im} f_{++}^j d_{\frac{1}{2}\frac{1}{2}}^j(\theta) \\ &= \frac{k}{4\pi} \left\{ \frac{1}{k^2} \sum_{j,j'} \left(j + \frac{1}{2} \right) \left(j' + \frac{1}{2} \right) f_{++}^j \cdot f_{++}^{j'*} \cdot \int d\Omega D_{\frac{1}{2}\frac{1}{2}}^j(\varphi, \theta', 0) \cdot D_{\frac{1}{2}\frac{1}{2}}^{j'}(\varphi', \theta'', \varphi'') + \right. \\ & \left. + \frac{1}{k^2} \sum_{j,j'} \left(j + \frac{1}{2} \right) \left(j' + \frac{1}{2} \right) f_{+-}^j \cdot f_{+-}^{j'*} \cdot \int d\Omega D_{\frac{1}{2}-\frac{1}{2}}^j(\varphi, \theta', 0) \cdot D_{\frac{1}{2}-\frac{1}{2}}^{j'}(\varphi', \theta'', \varphi'') + F_{in}^{++} \right\}. \end{aligned} \tag{8}$$

$$d\Omega = d \cos \theta' \cdot d\varphi \tag{9}$$

$$\cos \theta'' = \cos \theta \cdot \cos \theta' + \sin \theta \cdot \sin \theta' \cdot \cos \varphi \tag{10}$$

The angles $\varphi, \theta', \varphi', \theta'', \varphi''$ are the Euler angles for rotation around successive axes presented in Fig. 1 as required in the function $D(\alpha, \beta, \gamma)$. Detailed formulae for the elastic overlap functions in terms of the above angles are given in the Appendix.

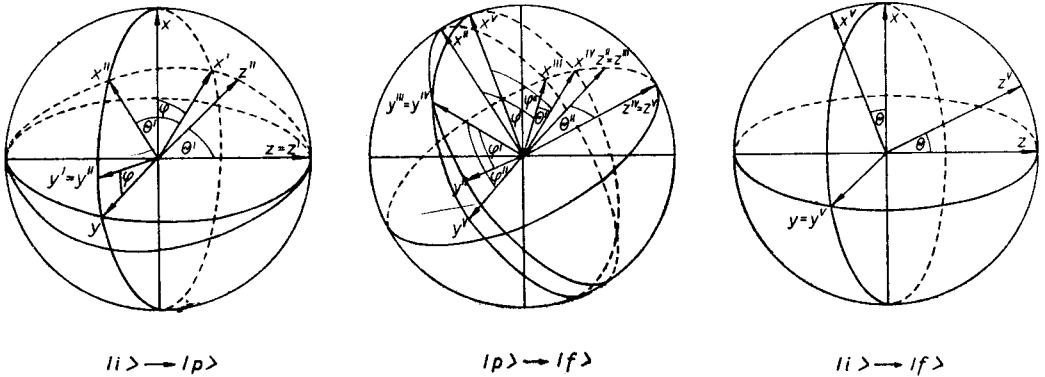


Fig. 1. The Euler rotation angles, which are needed to transform the initial state $|i\rangle$ to the intermediate state $|p\rangle$ and from $|p\rangle$ to the final state $|f\rangle$ of the πp case

After integration of (8), we obtain the helicity non-flip overlap function

$$F_{in}^{++} = \sum_j \frac{2\pi}{k^2} \left(j + \frac{1}{2} \right) (2 \text{Im} f_{++}^j - |f_{++}^j|^2 - |f_{+-}^j|^2) d_{\frac{1}{2}\frac{1}{2}}^j(\theta) = \sum_j \sigma_{in}^j d_{\frac{1}{2}\frac{1}{2}}^j(\theta) \tag{11}$$

which corresponds to equation (3) for the spinless case. Here, σ_{in}^j is the inelastic partial cross-section for the total angular momentum j , as may be shown by consideration of the probability conservation. It is, however, well known that in the pion-proton scattering not

only the total angular momentum, but also parity is conserved in the interaction. Hence, introducing the partial amplitudes for a definite j and parity:

$$f_{l,+} = \frac{1}{2} (f_{++}^j + f_{+-}^j) \quad j = l + \frac{1}{2} \quad (12a)$$

$$f_{l,-} = \frac{1}{2} (f_{++}^j - f_{+-}^j) \quad j = l - \frac{1}{2} \quad (12b)$$

one may define partial inelastic cross-sections for those values of j and parity:

$$\sigma_{in}^j = \sigma_{in}^{j,+} + \sigma_{in}^{j,-} \quad \text{and} \quad \sigma_{el}^j = \sigma_{el}^{j,+} + \sigma_{el}^{j,-} \quad (13)$$

These cross section are related to the elasticity (absorption parameter) η , commonly used in the phase-shift analysis, by the formula

$$\sigma_{in} = \frac{\pi}{k^2} \left(j + \frac{1}{2} \right) (1 - \eta^2) \quad (14)$$

Therefore, the unitarity equation (4a) for the partial waves may be written as

$$\frac{4\pi}{k^2} \left(j + \frac{1}{2} \right) \text{Im} f_{l,+} + \frac{4\pi}{k^2} \left(j + \frac{1}{2} \right) \text{Im} f_{l,-} = \sigma_{el}^{j,+} + \sigma_{el}^{j,-} + \sigma_{in}^{j,+} + \sigma_{in}^{j,-}. \quad (15)$$

After similar treatment of the second unitarity equation (4b) we get

$$F_{in}^{+-} = \sum_i (\sigma_{in}^{i,+} - \sigma_{in}^{i,-}) d_{\frac{1}{2}-\frac{1}{2}}^i(\theta) \quad (16)$$

and

$$\frac{4\pi}{k^2} \left(j + \frac{1}{2} \right) \text{Im} f_{l,+} - \frac{4\pi}{k^2} \left(j + \frac{1}{2} \right) \text{Im} f_{l,-} = \sigma_{el}^{j,+} - \sigma_{el}^{j,-} + \sigma_{in}^{j,+} - \sigma_{in}^{j,-}. \quad (17)$$

The linear combinations of (15) and (17) give us the well known unitarity equations for the partial waves with defined parities

$$\frac{4\pi}{k^2} \left(j + \frac{1}{2} \right) \text{Im} f_{l,+} = \sigma_{el}^{j,+} + \sigma_{in}^{j,+} \quad (18a)$$

$$\frac{4\pi}{k^2} \left(j + \frac{1}{2} \right) \text{Im} f_{l,-} = \sigma_{el}^{j,-} + \sigma_{in}^{j,-} \quad (18b)$$

They have been extensively discussed by Henzi [2], [3], who studied a model of high-energy πp scattering in terms of the inelastic partial cross-sections (denoted as g_l and \bar{g}_l) expressed as functions of impact parameter b .

3. Experimental properties of the overlap functions in the πp scattering

From the knowledge of the elastic scattering amplitudes some general information about inelastic πp reactions can now be deduced. Namely, we may compute both the inelastic overlap functions

$$F_{in}^{++} = \frac{4\pi}{k} \text{Im} f_{++} - F_{el}^{++,+} - F_{el}^{+,-,+} \quad (19a)$$

and

$$F_{in}^{+-} = \frac{4\pi}{k} \text{Im} f_{+-} - F_{el}^{++,-+} - F_{el}^{+,-,-} \quad (19b)$$

and from them, using (11), (13) and (16), derive all the inelastic partial cross-sections with definite angular momenta and parities. This can be useful, because at present it seems impossible to determine the inelastic overlap functions (or partial cross-sections) directly from measurements of inelastic channels [4].

Although the amplitudes are not directly measurable, there exist methods of deriving them from the available experimental data. The most reliable is the phase-shift analysis which at present is performed up to about 2 GeV/c pion laboratory momentum. At higher momenta various models may be applied to reconstruct the amplitudes from measurements of differential cross-sections and polarization with the use of some special assumptions, like *e.g.* the Regge-exchange hypothesis.

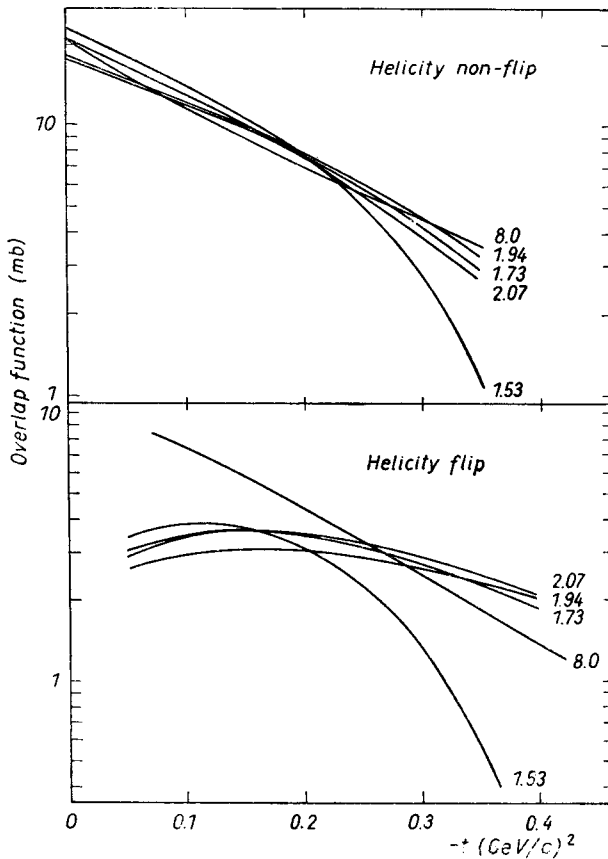


Fig. 2. The four-momentum transfer dependence of the helicity non-flip F_{in}^{++} and helicity flip F_{in}^{+-} overlap functions for inelastic processes, computed for low energies from the phase-shift analysis and for 8 GeV/c from the multi-Regge-exchange model

In the present paper we have computed the overlap functions F_{in}^+ for 1.53, 1.73, 1.94, 2.07 GeV/c π^+p scattering from the phase-shift analysis of Lovelace *et al.* [5] and for 8 GeV/c from the model of Rarita *et al.* [6]. They are shown in Fig. 2 and the exponential slopes of the overlap functions F_{in}^{++} are presented in Table I.

TABLE I

P_{lab} [GeV/c]	1.53	1.73	1.94	2.07	8.0
A slope of F_{in}^{++} $0 < -t < 0.15$ (GeV/c) ²	5.05	4.12	4.17	4.75	5.45
R [fermi]	0.627	0.566	0.569	0.608	0.651

It may be noted that the slope of F_{in}^{++} which is connected with the region of interaction¹ does not change much with the increase in the pion laboratory momentum from 1.5 to 8.0 GeV/c.

In this work the overlap functions for 8 GeV/c π^+p elastic scattering have been calculated with appropriate treatment of spin and variable phase effects, as prescribed by the Regge-pole model. The slope $A = 2.71$ (GeV/c)⁻² of the obtained overlap function $F_{el}^{++} = F_{el}^{+,+,+} + F_{el}^{+,-,+}$ may be compared with the slope $A = 1.84$ (GeV/c)⁻² of the overlap function F_{el}^+ , calculated as if it were spinless particles, from the scalar amplitude taken as $\sqrt{\frac{d\sigma}{d\cos\theta}}$. (The dependence of the phase on the scattering angle has been neglected in the latter calculation).

The problem of the influence of spin and phase on the overlap function calculation is important when one considers unitarity requirements for two- and many-body reactions at high energies following from the knowledge of elastic scattering differential cross-section (see *e. g.* Ref. [8]).

It is known experimentally that two-body inelastic reactions have differential cross-sections which may be usually described as exponential functions of four-momentum transfers t , with slopes of the same magnitude as for the elastic scattering [9], [10].

The unitarity would then require that the overlap functions for many-body channels are steeper than those for two-body channels, and therefore many-body reactions would be more peripheral². This conclusion is however not valid if the spin or phase effects are important in the calculation of the overlap function. In order to check this one needs reliable models of two-body reactions which would yield a complete knowledge of the amplitudes. The present result for the elastic scattering may be considered as an example, in which the discussed effects are not so important.

¹ For the Gaussian distribution of the incident wave absorption: $\sigma_{in}^j \sim \left(j + \frac{1}{2}\right) \exp\left(-\frac{j^2}{2R^2k^2}\right)$, with

the dispersion R , one may show [7] that $R = \sqrt{2A}$.

² There has already been a hint [11] that this may happen in the multi-Regge-exchange model of inelastic collisions.

4. Conclusions

Two overlap functions F_{in}^{++} and F_{in}^{+-} have been introduced to represent the inelastic partial cross-sections summed over all channels. The slope of the non helicity-flip overlap function F_{in}^{++} practically does not depend on energy in the range from 1.5 to 8 GeV/c, this slope being the measure of the interaction radius. The helicity flip overlap function F_{in}^{+-} represents differences in cross-sections for the same angular momentum and opposite parities.

The author wishes to express his gratitude to Dr L. Michejda, who introduced him to this subject, for constant help, encouragement and many illuminating discussions. The author is also grateful to Dr A. Białas for valuable remarks on the manuscript.

APPENDIX

Calculation of the elastic overlap functions from the scattering amplitudes.

$$F_{el}^{++++} = \int d\Omega f_{++}^{i \rightarrow p} \cdot (f_{++}^{f \rightarrow p})^* = \int d\Omega f_{++}(\theta') e^{\frac{i}{2}\varphi} \cdot f_{++}^*(\theta'') e^{-\frac{i}{2}(\varphi'+\varphi'')}$$

Taking advantage of (9), (10) and

$$e^{i\varphi'} = \frac{\cos \theta \sin \theta' \cos \varphi - \sin \theta \cos \theta' + i \sin \theta' \sin \varphi}{\sin \theta''} e^{\frac{i}{2}(\varphi'+\varphi'')}$$

$$= \frac{\cos \frac{\varphi}{2} \left(\cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right) + i \left(\cos \frac{\theta}{2} \cos \frac{\theta'}{2} - \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right) \sin \frac{\varphi}{2}}{\cos \frac{\theta''}{2}}$$

we get

$$F_{el}^{++++} = \int_{-1}^1 d \cos \theta' \int_0^{2\pi} d\varphi \left\{ [\operatorname{Re} f_{++}(\theta') \cdot \operatorname{Re} f_{++}(\theta'') + \operatorname{Im} f_{++}(\theta') \cdot \operatorname{Im} f_{++}(\theta'')] \times \right. \\ \times \frac{\cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} + \sin \frac{\theta}{2} \cdot \sin \frac{\theta'}{2} \cdot \cos \varphi}{\cos \frac{\theta''}{2}} - [\operatorname{Im} f_{++}(\theta') \cdot \operatorname{Re} f_{++}(\theta'') - \operatorname{Re} f_{++}(\theta') \times \\ \left. \times \operatorname{Im} f_{++}(\theta'')] \times \frac{\sin \frac{\theta}{2} \sin \frac{\theta'}{2} \cdot \sin \varphi}{\cos \frac{\theta''}{2}} \right\}$$

Similarly one can show that

$$F_{el}^{+-,+} = \int_{-1}^1 d \cos \theta' \int_0^{2\pi} d\varphi \left\{ [\operatorname{Re} f_{+-}(\theta') \cdot \operatorname{Re} f_{+-}(\theta'') + \operatorname{Im} f_{+-}(\theta') \cdot \operatorname{Im} f_{+-}(\theta'')] \times \right. \\ \times \left(\cos \frac{\varphi}{2} \cos \frac{\varphi' - \varphi''}{2} + \sin \frac{\varphi}{2} \sin \frac{\varphi' - \varphi''}{2} \right) - [\operatorname{Im} f_{+-}(\theta') \cdot \operatorname{Re} f_{+-}(\theta'') - \operatorname{Re} f_{+-}(\theta') \times \\ \times \operatorname{Im} f_{+-}(\theta'')] \times \left(\sin \frac{\varphi}{2} \cos \frac{\varphi' - \varphi''}{2} - \cos \frac{\varphi}{2} \sin \frac{\varphi' - \varphi''}{2} \right) \left. \right\},$$

where

$$\cos \frac{\varphi' - \varphi''}{2} = \frac{\sin \theta' \cdot \sin \varphi \left(\cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} - \sin \frac{\theta}{2} \cdot \sin \frac{\theta'}{2} \right) \cdot \sin \frac{\varphi}{2}}{\sin \theta'' \cdot \cos \frac{\theta''}{2}} - \\ - \left(\cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} + \sin \frac{\theta}{2} \cdot \sin \frac{\theta'}{2} \right) \times \\ \times \frac{\sin \theta' \cdot \cos \theta \cdot \cos \varphi - \sin \theta \cdot \cos \theta'}{\sin \theta'' \cdot \cos \frac{\theta''}{2}} \cdot \cos \frac{\varphi}{2} \\ \sin \frac{\varphi' - \varphi''}{2} = \frac{\sin \theta' \cdot \sin \varphi \cdot \left(\cos \frac{\theta}{2} \cos \frac{\theta'}{2} + \sin \frac{\theta}{2} \cdot \sin \frac{\theta'}{2} \right) \cdot \cos \frac{\varphi}{2}}{\sin \theta'' \cdot \cos \frac{\theta''}{2}} - \\ - \frac{\sin \theta' \cdot \cos \theta \cdot \cos \varphi - \sin \theta \cdot \cos \theta'}{\sin \theta'' \cdot \cos \frac{\theta''}{2}} \times \\ \times \cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} - \sin \frac{\theta}{2} \cdot \sin \frac{\theta'}{2} \left. \right\} \cdot \sin \frac{\varphi}{2}$$

and in a very similar way

$$F_{el}^{++,-} = \int_{-1}^1 d \cos \theta' \cdot \int_0^{2\pi} d\varphi \left\{ [\operatorname{Re} f_{++}(\theta'') \cdot \operatorname{Im} f_{++}(\theta') - \operatorname{Re} f_{++}(\theta') \cdot \operatorname{Im} f_{++}(\theta'')] \times \right. \\ \times \left(\sin \frac{\varphi}{2} \cdot \cos \frac{\varphi' - \varphi''}{2} + \cos \frac{\varphi}{2} \cdot \sin \frac{\varphi' - \varphi''}{2} \right) - \\ - \left(\cos \frac{\varphi}{2} \cdot \cos \frac{\varphi' - \varphi''}{2} - \sin \frac{\varphi}{2} \cdot \sin \frac{\varphi' - \varphi''}{2} \right) \times \\ \times [\operatorname{Re} f_{++}(\theta') \cdot \operatorname{Re} f_{++}(\theta'') + \operatorname{Im} f_{++}(\theta') \cdot \operatorname{Im} f_{++}(\theta'')] \left. \right\}.$$

$$F_{el}^{+-,--} = \int_{-1}^1 d \cos \theta' \cdot \int_0^{2\pi} d\varphi \left\{ [\operatorname{Re} f_{+-}(\theta') \cdot \operatorname{Re} f_{++}(\theta'') + \operatorname{Im} f_{+-}(\theta') \cdot \operatorname{Im} f_{++}(\theta'')] \times \right. \\ \times \frac{\cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} \cdot \cos \varphi + \sin \frac{\theta}{2} \cdot \sin \frac{\theta'}{2}}{\cos \frac{\theta''}{2}} + \frac{\cos \frac{\theta}{2} \cdot \cos \frac{\theta'}{2} \cdot \sin \varphi}{\cos \frac{\theta''}{2}} \times \\ \left. [\operatorname{Re} f_{+-}(\theta') \cdot \operatorname{Im} f_{++}(\theta'') - \operatorname{Im} f_{+-}(\theta') \cdot \operatorname{Re} f_{++}(\theta'')] \right\}.$$

REFERENCES

- [1] L. Van Hove, *Nuovo Cimento*, **28**, 798 (1963).
- [2] R. Henzi, *Nuovo Cimento*, **52A**, 772 (1967).
- [3] R. Henzi, *Polarization in the Overlap Function Analysis of π^+p Elastic and Charge Exchange Scattering at High Energies*, CERN preprint TH. 874 (1968).
- [4] L. Michejda, *Institute of Nuclear Research, Warsaw, preprint "P" 889/VI/PH*. and in print in "Fortschritte der Physik".
- [5] C. Lovelace, *Proceedings of the Heidelberg International Conference on Elementary Particles*, 1968.
- [6] W. Rarita, R. J. Riddell, C. B. Chiu, R. J. N. Phillips, *Phys. Rev.*, **165**, 1615 (1968).
- [7] L. Van Hove, *Theoretical Problems in Strong Interactions at High Energies*. Lectures given at CERN in 1964.
- [8] A. Białas, L. Van Hove, *Nuovo Cimento*, **38**, 1385 (1965).
- [9] Aachen — Berlin — Birmingham — Bonn — Hamburg — London — München Collaboration, *Nuovo Cimento*, **31**, 729 (1964).
- [10] Aachen — Berlin — CERN Collaboration, *Phys. Letters*, **19**, 608 (1965).
- [11] L. Michejda, J. Turnau, A. Białas, *Nuovo Cimento*, **56A**, 241 (1968).