# ELEMENTS OF THE ABERRATION MATRIX OF A THIN LENS LIMITED BY TWO SPHERICAL SURFACES

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The translation and refraction matrices of a thin lens have been presented, and the corresponding aberration matrix calculated. Also the connection between the elements of the aberration matrix and the quantities specifying the optical properties of the thin lens has been given.

In this paper an application of the aberration matrix (defined as a combination of Herzberger's translation and refraction matrices [1]) to a thin lens is shown. Ray tracing from an axial object point through the refracting surfaces of a single lens to the image plane can be described by three translation and two refraction matrices [2], [3]. By multiplying the five matrices in this way

$$Q = \mathcal{T}_3 \mathcal{R}_2 \mathcal{T}_2 \mathcal{R}_1 \mathcal{T}_1, \tag{1}$$

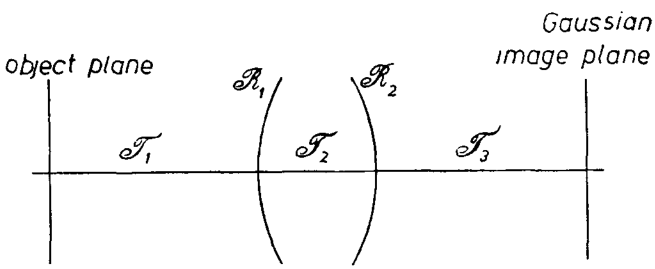


Fig. 1. Translation and refraction matrices of a single lens

we obtain a matrix which we call the matrix of the lens (generally: matrix of the optical system). By $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ we denote the translation matrices of the spaces between the object plane and the first refracting surface, between the first and the second refracting surface and between the second refracting surface and the Gaussian image plane, respectively (Fig. 1).

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By $\mathcal{R}_1$, $\mathcal{R}_2$ we denote the refraction matrices of the first and the second refraction surface, respectively. If the lens which we consider is a very thin lens ($d = 0$), then the matrix

$$\mathcal{T}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

However, the other matrices have the form

$$\mathcal{T}_1 = \begin{pmatrix} 1 & -\frac{z}{\zeta} \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{T}_3 = \begin{pmatrix} 1 & \frac{z'}{\zeta'} \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{R}_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix},$$

$$\mathcal{R}_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}, \tag{3}$$

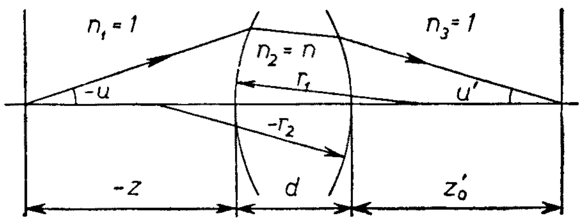


Fig. 2. An axial ray tracing through a single enss

where

$$\zeta = n_1 \cos u,$$

$$\zeta' = n_3 \cos u' \tag{4}$$

are the optical direction cosines of the rays in the object and image space respectively. The elements of the two refraction matrices are given by the following equations [1]

$$\alpha = \frac{1+\psi(z^*+r)}{1+r\psi},$$

$$\beta = \frac{z^*\psi(z^*+r)}{(1+r\psi)\zeta},$$

$$\gamma = -\psi\zeta,$$

$$\delta = 1 - z^*\psi, \tag{5}$$

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where

$$\psi = \frac{n' \cos i' - n \cos i}{r\zeta}$$

$z^*$ — axis coordinate of the intersection point of the ray with the refracting surface.

We can write the matrix (1) of a single lens in the general form

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \tag{1'}$$

where

$$q_{11} = \frac{z'}{\zeta'}(\alpha_1\gamma_2+\gamma_1\delta_2)+\alpha_1\alpha_2+\gamma_1\beta_2,$$

$$q_{12} = \frac{z'}{\zeta'}(\beta_1\gamma_2+\delta_1\delta_2) - \frac{z}{\zeta}(\alpha_1\alpha_2+\gamma_1\beta_2) - \frac{zz'}{\zeta\zeta'}(\alpha_1\gamma_2+\gamma_1\delta_2)+\beta_1\alpha_2+\delta_1\beta_2, \tag{6}$$

$$q_{21} = \alpha_1\gamma_2+\gamma_1\delta_2,$$

$$q_{22} = -\frac{z}{\zeta}(\alpha_1\gamma_2+\gamma_1\delta_2)+\beta_1\gamma_2+\delta_1\delta_2.$$

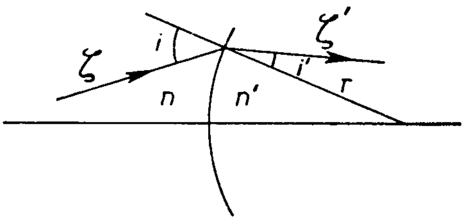


**Fig. 3. Ray refraction on the spherical surface**

The rays that are traced in the paraxial region form a perfect image in the Gaussian image plane, and the translation and refraction matrices of this region are for a single lens as follows:

$$\mathcal{T}_1^0 = \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_3^0 = \begin{pmatrix} 1 & z_0' \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{R}_1^0 = \begin{pmatrix} 1 & 0 \\ -\frac{n-1}{r_1} & 1 \end{pmatrix}, \quad \mathcal{R}_2^0 = \begin{pmatrix} 1 & 0 \\ \frac{n-1}{r_2} & 1 \end{pmatrix}. \tag{7}$$

By multiplying the matrices in the same way as in equation (1), we obtain the lens matrix of the paraxial region, the elements of which do not change but are always constant.

$$q_{11}^0 = 1 - z_0'(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$q_{12}^0 = z_0' - z + z_0' z(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

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$$q_{21}^0 = -(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$q_{22}^0 = 1+z(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{8}$$

We see that the said equations determine the optical properties of a thin lens. The element $q_{12}^0 = 0$, as

$$\frac{1}{z_0'} - \frac{1}{z} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{9}$$

Denoting by $f$ the focal length of the lens, the lens matrix of a paraxial region has a form

$$Q^0 = \begin{pmatrix} 1 - \frac{z_0'}{f} & 0 \\ -\frac{1}{f} & 1 + \frac{z}{f} \end{pmatrix}. \tag{10}$$

According to the definition [1] of the aberration matrix

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{11}$$

where $a_{11} = q_{11} - q_{11}^0$, $a_{12} = q_{12} - q_{12}^0$, $a_{21} = q_{21} - q_{21}^0$, $a_{22} = q_{22} - q_{22}^0$, we have

$$a_{11} = q_{11} + \frac{z_0'}{f} - 1$$

$$a_{12} = \frac{z_0'}{\zeta'} q_{22} - \frac{z}{\zeta} q_{11} + \frac{zz_0'}{\zeta\zeta'} q_{21} + \beta_1\alpha_2 + \delta_1\beta_2,$$

$$a_{21} = q_{21} + \frac{1}{f},$$

$$a_{22} = q_{22} - \frac{z}{f} - 1. \tag{12}$$

The element $a_{11}$ represents the difference of magnifications of a wide pencil aperture and of the paraxial rays [1]. When spherical aberration is absent, equation $a_{11} = 0$ determines Abbe's sine condition. $a_{22}$ expresses the absolute value of the element $a_{11}$ divided by the product of Gaussian and zonal magnifications. $a_{12}$ expresses the transverse spherical aberration in the Gaussian image plane for a numerical aperture equal to one. Finally the element $a_{21}$ expresses the difference between the power and the skew power of the lens.

## REFERENCES

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